A Measure Differential Inclusion Approach to Rigid Bodies Impacts

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Abstract: - A distinctive feature of the bodies collision study is the transition from the relative motion to a closed contact (capture) through a sequence of shocks, i.e., an impact accumulation point. This transition is automatically treated by time-stepping algorithms and, therefore, this type of numerical scheme will be employed in the present study. The dynamics of a system with impacts is completely described by an equality of measures and a set-valued impact law. The equation system forms a measure differential inclusion. Both impulsive and non-impulsive dynamics are casted in the same framework. It is also within the scope of this work to present practical solutions to the numerical instabilities which can occur when solving such systems. An application example regarding longitudinal train dynamics is presented.

Key-Words: - time-stepping algorithm, non-smooth dynamics, measure differential inclusion, longitudinal train dynamics

1 Introduction

The evolution of the natural or technical systems is, in most cases, subjected to restrictions. The most common constraints are perceived as geometric limitations as, for instance, imposing the movement along a specific curve for a rigid body or require two bodies to remain a specified distance apart. The study of the constrained dynamics aims to connect Newton’s laws and the geometric restrictions.

The first approach of the problem provided an approximate solution based on energy functions. Restrictions were imposed through rigid springs which accomplish a rough model of the constraint. In order to maintain a constraint, the spring constant should be large enough to produce large forces for small displacements. Such unbalance in the system parameters gives rise to stiff differential equations which are difficult or even impossible to integrate, see for instance [1].

Even more, the employment of additional energy in the system, known as penalty method, relates constraint forces with displacements. The displacements produced by applied forces act as signals that tell the constraint what restoring force is required. Therefore, apart from the stiffness introduced by the penalty constraints the system states will become subject to an erroneous drift, e.g., [1-3].

The alternative solution is to compute the forces required to maintain the constraints imposed, rather than using displacements and restoring forces. The role of these forces is to cancel the components of the applied forces which act against the constraints. Since forces produce acceleration, at this level constraint forces will assure the consistency between constraints and accelerations.

However, in order to follow this approach a series of problems has to be solved. In the rigid body assumption, constraint forces, i.e., normal or dry friction forces, are set-valued and do not fit in potential theory. Therefore, they have to be determined considering their relation with the motion of the dynamical system [4]. A consequence of this approach is that phase transition, i.e., constraints enabling and disabling, and contact forces activation brings discontinuities in the system evolution. While these discontinuities occur at acceleration level, impulsive forces result in discontinuities at velocity level.

A major problem of the calculation methods applied to solve non-smooth systems is the uniqueness and existence of the solution. Indeterminacy and inconsistency have been observed in many planar rigid body problems, in the presence of dry friction. Some of them are known as “Painleve paradoxes”. The problem comes to a
climax when the static friction forces (or other constraint types) are redundant. In this case, indeterminacy is attended by a rank-deficient constraint matrix.

Several computational methods have been developed, beginning with the regularization method which is completely determinate due to the “hard springs” added but brings the previously mentioned shortcomings. Fast and accurate methods using conditional statements to describe various modes of the system, as for instance the switch model [2], may not be used for a large number of contacts as the logical complexity increases in exponential rate.

Impulse velocity methods have been introduced and developed by [5]. Handling more than one impulse at a given moment was settled by [6]. The method proposed by [7] may even be used to model continuous contact. Also, in order to avoid inconsistent configurations, [8] solved the problem for impulsive forces and velocities (instead of forces and accelerations). Friction indeterminacy has also been approached as a probability problem by means of a statistical analysis of perturbed simulation results.

Convex analysis tools provide a general framework for problems involving set-valued laws. Inclusions may be solved using variational formulations, projections, proximations and associated optimization problems. Solutions are usually obtained iteratively solving projective equations by Jacobi or Gauss-Seidel algorithms, by quadratic programming or by solving complementarity problems. Detailed references may be found in [1, 4, 9].

In contact dynamics, models able to handle unilateral constraints have been extensively formulated as linear complementarity problem (LCP), e.g., [10]. In the case of friction forces, complementarity constraints ensure that either static or kinetic friction is applied. LCP solvers take advantage of the ample research works in mathematical programming. The point is that they allow the forces resolution through the complementarity equations and not by analyzing conditional statements.

It is emphasized, yet, that complementarity conditions and rigid body assumption confer a coherent mathematical framework but they may be still empirically corrected to improve their physical realism [5, 11]. As available algorithms to solve complementarity problem based contact models are mainly linear, the friction cone has to be linearized by means of some special arrangement of constraints. A poor approximation is to the detriment of accuracy while a better one is computationally more expensive. Still, the friction forces negate the LCP convexity and the well-posedness of the problem may be assured, but only for small values of the friction coefficients. In a study regarding the validity of the complementarity conditions, [11] recalls possible physical inaccuracies behind the complementarity assumption and recounts that many authors [5, 8, 10] are aware of them.

It may thus be inferred that adequate methods should be tailored for each application [12]. Mathematically equivalent problems may be formulated in order to follow different computational procedures, e.g., the LCP formulated in the contact space.

In this paper, a general framework for impulsive and non-impulsive dynamics is developed using an equality of measures formulated at the velocity level. The equations of motion are integrated using a Moreau time-stepping algorithm. Impulsive and normal contact forces are described by a set-valued law of Signorini type, while friction forces are described by a set-valued law of Coulomb type. An application example regarding longitudinal train dynamics is presented.

2 Newton restitution law

The classical Newton kinematic impact law used to predict the outcome of a simple collision, involves two bodies which have the masses \( M_i \) and \( M_{i+1} \); \( u^- \) and \( u^+ \) are the velocities before and after the collision, respectively, Equation (1).

\[
\begin{align}
M_i (u_i^+ - u_i^-) + M_{i+1} (u_{i+1}^- - u_{i+1}^-) &= 0 \quad (1a) \\
\dot{g}^-_i - e \dot{g}^+_i &= 0 \quad (1b)
\end{align}
\]

The above system of equation comprises the law of the conservation of momentum (1a), and the definition of the coefficient of restitution (COR), whose usual value is \( 0 \leq e \leq 1 \), (1b). The zero magnitude corresponds to a perfectly inelastic collision, at which the relative velocity \( \dot{g}^-_i = u_i^+ - u_{i+1}^- \) vanishes, whereas the magnitude 1 represents a perfectly elastic one.

Moreover, the impact equation which describes the impulsive motion may be formulated introducing the impulsive forces \( P \) as in Equation (2)

\[
M (u^+ - u^-) - W P = 0 \quad (2)
\]
where \( M \in \mathbb{R}^{n \times n} \) is the inertia matrix, \( M = diag(M_i), \ i \in \{1, \ldots, n\} \), and \( W_p \in \mathbb{R}^{n \times m} \) the corresponding constraint matrix which describes the action of the impulsive forces.

Impulsive forces, \( p \), can be described by set-valued laws [5]. When two bodies close in and the gap \( g \) is covered, a rebound may take place and the relative velocity would become positive. The forces involved in common collisions act as compressive magnitudes which preclude the closing bodies from interpenetrating. Therefore, impulsive force may be defined as in Equation (3).

\[
P \in \begin{cases} [0, \infty), & \text{if } g = 0 \land \dot{g} < 0, \\ 0, & \text{else} \end{cases} \quad (3)
\]

An impact occurs between two bodies which come into contact and the relative velocity before the impact does not vanish.

The set-valuedness of the impulsive forces may yield indefinite values. However, the use of the general Newton restitution law, Equation (1), always provides a solution.

### 3. The dynamics with impacts problem

The classical Newton-Euler equations which describe unconstrained motion of a multibody system with \( n \) degrees of freedom may be stated as:

\[
M \ddot{q} - h(t, q, \dot{q}, u) = 0 \quad (4a)
\]

\[
\dot{q} = u \quad (4b)
\]

where \( q \in \mathbb{R}^n \) is the generalized coordinates vector, \( u \in \mathbb{R}^n \) the generalized velocities vector and \( h \in \mathbb{R}^n \) is the sum of the forces described by constitutive laws – the inertia matrix \( M \).

Further, the impact equation which describes the impulsive motion may be formulated introducing the impulsive forces \( p \) as in Equation (2). Using an equality of measures [4], the impulsive and non-impulsive dynamics may be embraced in a general formulation. The motion and impact equations (4) and (2) may be multiplied with the Lebesgue measure and the atomic measure (i.e., the sum of the Dirac point measures at the collisions) and they may be added to yield Equation (5). In this framework, the numerical solution may be accomplished within a unitary integration process [12].

\[
M[\ddot{u}dt + (u^+ - u^-)d\eta] - hdt - W_p\dot{p}d\eta = 0 \quad (5)
\]

Constitutive forces laws are included in the vector \( h \). The dynamics of a system with impacts is completely described by the equality of measures (5) and by the set-valued law of the impact (3). The equation system forms a measure differential inclusion which may be solved using time-stepping methods. Each integration step requires the resolution of the impulsive forces.

### 3.1. The time-stepping algorithm

A distinctive feature of the bodies collision study is the transition from the relative motion to a closed contact (capture) through a sequence of shocks, i.e., an impact accumulation point. This transition is automatically treated by time-stepping algorithms and, therefore, this type of numerical scheme will be employed in the study.

Time-stepping methods are most simple and robust [2, 13, 14]. Moreover, they are a real difference equations approach and they are based on velocity constraints, thus reducing the system index.

In the present work, the Non-smooth Contact Dynamics (NSCD) method [4] will be considered. The numerical scheme is very general and is proved to converge globally for \( 0 \leq e \leq 1 \), [15]. The drawback of the method is that it uses a first order discretization, but in the case of the studied case, this is not relevant as the system undergoes many collisions and after each impact, the order decreases to one [16].

The integration step begins with the computation of the midpoint time instant \( t_m \) and the corresponding position state vector and constitutive forces. Considering that the collision is infinitesimal while resting contacts last for the whole simulation step, the impacts resolution is accomplished before the constitutive forces computation. Finally, the state vector is updated. Index \( m \) is used for the initial values of the integration step and \( b \) for the final values. Considering that the length of the step is \( \Delta t \) the discretization of (5) comprises the following stages.

1. Computation of the midpoint time instant \( t_m \), the corresponding position state vector \( q_m \) and the constitutive forces vector \( h_m \).

\[
t_m = t_a + 1/2\Delta t \quad (6)
\]

\[
q_m = q_a + 1/2u_a\Delta t \\
h_m = h(t_m, q_m, u_a)
\]

2. Collision detection and impulse resolution, which is accomplished analyzing the vectors \( g \) and \( g \) as indicated in (3).
3. Update the state vector.

\[
\begin{align*}
    \mathbf{u}_b &= \mathbf{u}_a + M^{-1}(h_m\Delta t + W P) \\
    \mathbf{q}_b &= \mathbf{q}_m + 1/2\mathbf{u}_b\Delta t
\end{align*}
\]  

(7)

3.2 Impulsive forces resolution

The constraint matrices relate generalized coordinates and forces to the contact space. The gap and the relative velocity in the contact points are consequently given by (8).

\[
\begin{align*}
    \mathbf{g} &= W^T_p \mathbf{q}_m & (8a) \\
    \dot{\mathbf{g}} &= W^T_p \dot{\mathbf{q}}_m & (8b)
\end{align*}
\]

The detection of the impacts and phase transitions instants is one of the problems related to the simulation of systems with collisions. A review of the accurate methods to locate zero-crossings may be found in [17], but, as a general feature, they are very expensive in terms of the required computational power. Besides, machine epsilon impedes maintaining numerically the zero level. A simple solution is given by the Karnopp model [16] which formulates the zero velocity condition as \( |\dot{\mathbf{q}}| < \eta \) (instead of \( \dot{\mathbf{q}} = 0 \)).

As explained above, the collision resolution precedes the constitutive forces computation. In the next stage of each integration step, possible impacts are sought verifying the conditions expressed in (3).

4 Solutions to numerical instabilities

Due to the system discontinuities numerical instabilities can occur[2]. However, because of its desirable properties, this technique is used to obtain all simulation results presented in this paper.

A widely used stabilization technique is the Baumgarte method, e.g., [21], which replaces the constraint acceleration equation with a linear combination of the acceleration, velocity and position. The physical interpretation of this method is that correction forces are added to counteract drift [22] in direct proportion to the error in the velocity and position constraints. The equation of GI constraint accelerations introducing the Baumgarte stabilization and the resulting improvement of numerical efficiency is analyzed in [23].

In the case of static friction the position is irrelevant for the numerical stability of the stick phase and the acceleration constraint depends only on the relative velocity. Improving the Karnopp model, [2] introduced a linear term in the static friction force that forces the relative velocity to converge to zero. The linear term eliminates possible chattering at the boundary of the zero velocity intervals.

A further study, namely [24], proposes an alternative friction model which uses the width of the stick band to join the value of kinetic friction to the static value, removing the discontinuity caused by the slip to stick switch. Compared to the previous methods, the latter one adds a non-linear term which can be determined from the continuity condition imposed on the friction force.

In practice, \( \eta \) should be much lower than velocity amplitude to avoid any possible influence upon the solution and solver tolerance. Thus, \( Tol \) should be much lower than \( \eta \); \( \max (\dot{x}) \gg \eta \gg Tol \). The lower limit of \( Tol \) and \( \eta \) is directly related to the efficiency of the numerical method; if an extremely tight tolerance is imposed, an unjustified increase in CPU time will be recorded. The method presented in [24] produces very good results even for very large values of \( \eta \) and considerably improves numerical efficiency. However, there is no general
method for the numerical stabilization of constrained systems integrated with standard ODE solvers and an optimum method should be sought for each particular model [1, 25].

Techniques of improving the numerical scheme to obtain a numerically stable solution are extensively discussed in [26]. Heuristic scaling of the system model parameters may also lead to very good results.

An application example of time-stepping integration is illustrated for the example described in [18]. The demonstrative application depicts a six vehicle train closing to a 7th resting braked vehicle. The simulation starts when the train leading vehicle (virtually, the train engine) applies instantly the entire breaking force. Five seconds later, the resting vehicle is collided. In about 15 seconds, all the vehicles come to rest.

Consistent integration and heuristic parameter scaling make possible to obtain a satisfactory stable integration. Relative displacement time history and contact forces are plotted in Figures 1 and 2. Captures and closed contacts can be noticed in the relative displacement plot.

5 Conclusions

In this paper, a general framework for impulsive and non-impulsive dynamics is developed using an equality of measures formulated at the velocity level. The equations of motion are integrated using a Moreau time-stepping algorithm. Impulsive and normal contact forces are described by a set-valued law of Signorini type, while friction forces are described by a set-valued law of Coulomb type. An application example regarding longitudinal train dynamics is presented.

Practical solutions to numerical instabilities are described. An application example proves the efficiency of the method.

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