An Event Driven Resolution of Driving Axles Stick-Slip

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Abstract: - As a consequence that the stick-slip motion phases involve different mechanisms, the differential equations which describe it are discontinuous. Such equations may be solved by means of a regularization (also called penalty) method which replaces the discontinuous equations by a smooth adjoint system. This solution is easy to follow as it consists of a common ODE system. In reverse, the equations are stiff, determining high integration costs and, the worst thing, not only the obtained values are inexact, but also the solution behavior. The alternative method to solve the problem is to deal with the nonlinear characteristic of the dry friction force. This choice is more suitable but also more difficult to apply. The non-smooth approach faces disjoint subintervals of the computation time. This approach involves special techniques to determine the switch moments, i.e., events, and the system mode. The solution proposed in this work aims to solve these issues by means of a non-smooth rolling friction model for driving wheels of railway vehicles stick-slip. A comparison with penalized approach and experimental data is performed. The solution features a marked higher numerical efficiency and accurate modeling of the phenomena.

Key-Words: - event-driven algorithm, non-smooth dynamics, set-valued friction law, driving axle, numerical efficiency

1 Introduction
As a consequence that the stick-slip motion phases involve different mechanisms, the differential equations which describe it are discontinuous. Such equations may be solved by means of a regularization (also called penalty) method which replaces the discontinuous equations by a smooth adjoint system. This solution is easy to follow as it consists of a common ODE system. In reverse, the equations are stiff, determining high integration costs and, the worst thing, not only the obtained values are inexact, but also the solution behavior. For instance, the occurrence of stick-slip may not be observed.

The alternative method to solve the problem is to deal with the nonlinear characteristic of the dry friction force. This choice is more suitable but also more difficult to apply. The non-smooth approach faces disjoint subintervals of the computation time. On each subinterval the system features a certain configuration of stick or slip phases between the bodies in contact which shall be further referred as system mode, while the phase’s switches will be referred as events.

2 Event-driven algorithms
The event-driven method considers the simulation interval as a union of disjoint subintervals on which the mode, stick or slip between the bodies in contact, remains unchanged. Until any of the contact phases switches, the system may be described by a set of ordinary equations; when at least one of the contacts phase changes, from sticking to relative motion or vice-versa, the corresponding friction
force law and, possibly, its value becomes different. Even more, stictions lead to DOF collapses. Consequently, on the following subinterval, the system might be described by a new set of equations and this is the case of the switching diagrams, [1, 2]. Such a tackling manner may not be pursued unless there are only a few contacts as the logical complexity of the model exponentially increases with the number of contacts. The alternative is to preserve the full slip dynamics system and to assign the proper values of the constraints. This solving manner involves special techniques to determine the moments of the switches, the system modes and the values of the static forces [3, 4].

The main disadvantage of the event-driven integration methods is that the logical complexity of the model exponentially increases with the number of friction contacts. As a consequence, they are not suitable for systems with many contact points [5] because, practically, the model may not be described. For instance, ten frictional contacts would imply over one thousand different equation systems (2^{10}, precisely).

In each mode, the dynamical system is represented by an ODE system that can be integrated by any standard solver. The transition between two modes is activated when the sign of a guard function changes, i.e., when an event is detected.

The algorithm includes the following stages [1]:
1. Determine the next smooth mode, initialize the system and update the equations.
2. Integrate the smooth state vector with any ODE solver while constraints are not violated.
3. Detect within imposed tolerance the moment of the next event.

In the implementation of this algorithm, two issues have to be solved: event detection and static friction computation [6].

2.1 Event detection.
In the case of systems with Coulomb friction models, transition from slip to stick may occur when relative velocity $\dot{\xi}$ vanishes in a contact point. The mode change is confirmed if the contact force lies strictly inside the friction cone $|\Sigma F| \leq T_{SMax}$, the limit value of the static friction force, also known as traction.

A review of the accurate methods to locate zero-crossings may be found in [1], but, as a general feature, they are very expensive in terms of the required computational power. A solution is given by the Karnopp model [7] which considers that the stick occurs when the relative velocity is “small enough”; this condition is formulated as $\dot{\xi} < \eta$, (instead of $\dot{\xi} = 0$), where $\eta$ should be much smaller than the average speed values of the system elements. The slip mode is defined by the complementary relative velocities which lie outside the narrow stick band, $\dot{\xi} < \eta$, (instead of $\dot{\xi} = 0$). As a result, one obtains a discontinuous system, non-stiff in the stick interval.

This method overcomes the problems of the zero velocity detection and allows efficient simulations; the stick band may be quite coarse but the stick and slip periods are nevertheless distinguished. On the other hand, because the relative acceleration is put to zero in the stick phase, the constant offset of the relative velocity causes a drift-off effect for large intervals and can cause numerical instability of the ordinary differential equations (ODE) integrator [2]. However, because of its desirable properties, this technique is used to obtain all simulation results presented in this paper.

In the stick mode, transitions may occur when the contact force reaches the boundary of the friction cone. This is accomplished when the static force equals $T_{SMax}$. Testing such an event requires root finding procedures.

2.2 Mode selection and integration of a smooth mode.
When events take place, the following system configuration has to be identified. Relative velocities and friction forces values grant the proper mode selection. The full slip dynamics may be integrated in a straight manner by any ODE solver. Sticking modes, however, need the computation of the static friction forces. In fact, event detection, mode selection and integration step depend on each other and may be simultaneously resolved. The core of the problem is to determine the static friction forces.

A major problem of the calculation methods applied to solve non-smooth systems is the uniqueness and existence of the solution. Indeterminacy and inconsistency have been observed in many planar rigid body problems. Some of them are known as “Painleve paradoxes”. Such a case arises when the number of the static friction forces is larger than that of the DOF.

Several computational methods have been developed, beginning with the penalty method which is completely determinate due to the “hard springs” added but brings the previously mentioned shortcomings. Fast and accurate methods using conditional statements to describe various modes of
the system, as for instance the switch model [1], may not be used for a large number of contacts as the logical complexity increases in exponential rate. In contact dynamics, friction forces have been extensively modelled by means of complementarity constraints which ensure that either static or kinetic friction is applied. The constraints take the form equation and define a linear complementarity problem (LCP) [8]. LCP solvers take advantage of the ample research works in mathematical programming. The point is that they allow the forces resolution through the complementarity equations and not by analyzing conditional statements.

It is emphasized, yet, that complementarity conditions and rigid body assumption confer a coherent mathematical framework but they may be still empirically corrected to improve their physical realism. As available algorithms to solve complementarity problem based contact models are mainly linear, the friction cone has to be linearized by means of some special arrangement of constraints. A poor approximation is to the detriment of accuracy while a better one is computationally more expensive. Still, the friction forces negate the LCP convexity and the well-posedness of the problem may be assured, but only for small values of the friction coefficients.

Also, in order to avoid inconsistent configurations, [9] solved the problem for impulsive forces and velocities (instead of forces and accelerations). It may be inferred, therefore, that adequate methods should be tailored for each application [1]. In the sequel, a complementarity-free model is presented.

3 A non-smooth friction model for rolling contacts

The rolling friction models feature the following characteristics: The friction law may encounter large variations due to rail condition. The experimental verification of the theory showed that high accuracy of the calculation of the creep forces is not required [10] and the simplified traction-displacement relation can be used in designing railway vehicles. The creepage values which correspond to unsaturated friction lie in a far more narrow range than those of the saturated friction.

While the first features allow the use of a more effective, approximate friction law in the study of the axle stick-slip, the latter points out the model stiffness. Stiff ordinary differential equations (ODE) are numerically unstable, unless the step size is taken to be extremely small and they are expensive in terms of computational power.

For a complex simulation of dynamic behaviour and control of locomotives or traction vehicles in connection with drive dynamics and traction control, the different creep force stages described above have to be assembled into one model [11]. The features requested by control models are to be sufficiently accurate, suitable for simulation (fast and easy to compute) and simple, keeping only essential parameters. Special models are needed if stiction occurs.

The solution proposed in this work aims to solve these issues by means of a non-smooth rolling friction model, Equation 1. Non-smooth models provide efficient and accurate solutions for rigid multibody systems [12], e.g., [13], a study of the switch model or [2], a complex non-smooth train dynamics research. Taking into account the characteristics of the phenomena, a set-valued friction law will be used to describe unsaturated friction, in a very similar way to the static friction set-valued models.

As within the scope of the model proposed is to account for the vehicles dynamics at start, i.e., when the vehicle velocity is zero or close to zero. Therefore, the usual measure of creep is replaced by the relative velocity in the wheel-rail contact point, \( \xi \). The main distinctive characteristic of this friction law is that, while the saturated friction force \( F_s \) is described by a “classical” constitutive law as one of those mentioned above, the adhesion force \( F_a \) can take any value in the set \([0 \ F_s \text{Max}]\) as long as \( \xi \) is bellow \( \eta_s \), the upper bound of the stick phase [12]. Due to the set-valuedness of the dynamics of the system is described by a differential inclusion [2].

\[
F_r \in \begin{cases} 
F_s(t, \xi), & \text{if } \xi > \eta_s, \text{(slip)} \\
F_a \in [0 \ F_s \text{Max}], & \text{if } \xi \leq \eta_s, \text{(stick)} 
\end{cases}
\] (1)

Since the stick-slip motion phases involve different mechanisms, the differential equations which describe it are discontinuous. Such a model is more reliable as it replaces stiff differential systems, but also more difficult to study [12, 14, 15]. The non-smooth approach has to deal with disjoint subintervals of the computation time. On each subinterval the system produces a certain configuration of stick or slip phases between the parts in contact, i.e., system mode.

This solution involves special techniques to determine the switch moments, i.e., events, and the system mode. The core problem is to determine the value of the adhesion force in the set \([0 \ F_s \text{Max}]\).
Another issue is related to the non-zero value of $\eta$ which can be derived solely from numerical considerations, although physical reasoning can also be employed [32]. All these aspects are elucidated in the following sections.

Structure experimental model: 1- driving motor, 5, 13- frame bogie, 5, 10- axle box, 8, 6- wheel lever download, 9, 4- wheel set, 10, 3 – gearing, 12, 7 – screw tightening, 2- drive axis.

On this bench, the normal force between wheels and beams can be adjusted using a lifting screw (12). Longitudinal and transversal forces can be measured using DC force sensors. The wheels loads are measured with a sensor (16) and the variable vertical contact force is measured using a piezoelectric force transducer (17, 18). The rotational speed is measured using both an infrared emitting tachometer (sensor nr. 15) and a wired angular sensor (3, 13). Vertical and longitudinal accelerations are measured using piezoelectric sensors (8, 9, 11, 12, 4). The input voltage from is fed into the AC asynchronous motor after several adjustments of the wheel load are performed to obtain stick-slip vibrations.

### 3.1 Description of the setup

The test bench considered in the experiment comprises two wheels connected by an axle (9, 4), see Figure 2. The only degree of freedom for each wheel is rotation and they can sleep on a supporting beam. The axle is driven by an electric motor (1), through an elastic drive shaft (2) and a gear (3). The gear is asymmetrically mounted being, in fact, adjacent to one of the wheels. The shaft is flexible in order to decouple the axle vibrations from the motor.

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### 3.2 Model of the setup

The aim of the setup model is to prove that the rolling friction non-smooth law is able to simulate properly the stick-slip of the driving axles. Therefore, only the elements which are essential for the system dynamics are taken into account.

The bodies which constitute the setup model are the wheels; their moments of inertia are $I_1$ and $I_2$, respectively, Figure 3. The only degree of freedom of each wheel is the rotation around its own axis; angular velocities are $\omega_1$ and $\omega_2$. The gap in the angular displacements of the wheels, $\theta$, produces an elastic torque, $M_{el}$. The engine torque, $M_m$, is applied to the first wheel and the second wheel is actuated by the elastic torque. Both wheels are also subjected to the friction forces and their resulting torques, $M_{f1}$ and $M_{f2}$. Hence, the stick-slip vibrations of the driven axle is described by Equations 2.

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**Figure 1.** Friction force in rolling contacts.

**Figure 2.** Experimental setup for the study of the powered axle

a- Experimental model, b- Simplified drawing of the experimental model.

**Figure 3.** A simple model for the axle torsional vibrations.
The values of the elastic and friction torque are given in Equations 5. The elastic torque is proportional to axle torsional stiffness $C$. The friction torque is proportional to the wheel radius, $R$.

According to Equations 1, in the case of adhesion the friction force is set-valued. The indefiniteness may be elucidated taking into account that in the stick phase the friction force is equal to the traction force. An additional condition has to be introduced to prevent the possibility of exceeding $F_{s\text{Max}}$. This condition also makes possible the transition from stick to slip in the numerical computation process.

\[
F_{f1} = \begin{cases} 
F_{s1}, & \text{if } \dot{\xi}_1 > \eta \\
\frac{M_1}{R}, & \text{if } \frac{M_1}{R} \leq F_{s\text{Max}} \\
F_{s\text{Max}}, & \text{if } \dot{\xi}_1 \leq \eta \text{ and } \frac{M_1}{R} > F_{s\text{Max}}
\end{cases}
\]

\[
F_{f2} = \begin{cases} 
F_{s2}, & \text{if } \dot{\xi}_2 > \eta \\
\frac{M_2}{R}, & \text{if } \dot{\xi}_2 \leq \eta \text{ and } \frac{M_2}{R} \leq F_{s\text{Max}} \\
F_{s\text{Max}}, & \text{if } \dot{\xi}_2 \leq \eta \text{ and } \frac{M_2}{R} > F_{s\text{Max}}
\end{cases}
\]

where the moments of the external forces are given by Equation 4:

\[
M_1 = M_m - C \cdot \theta \\
M_2 = C \cdot \theta
\]

As a consequence of the piecewise friction law given in Equations 3 the system described by Equations 2 becomes a switch model [27] which can be defined as follows.

In the switch model, the logical complexity exponentially increases with the number of contact points, e.g., [15] and the evaluation of the inequalities (inclusions) may prove to be difficult. However, as the present system has only two friction contacts (and due to its particular features) its inequalities may be evaluated without involving any special techniques. The solution involves spectacular mathematical features as the equations become a differential algebraic system and degrees of freedom collapse during stick phases.

And improved version of the switch models which analyses the stick phases of a system distinguishing between transitions and attractive or repulsive sliding modes has been proposed by [12]. In order to overcome the numerical instabilities involved by this method in the stick phase, in the case of a continuous stick (attractive mode), the system is “guided” to the middle of the stick band, where the relative velocity is exactly zero. This also prevents numerical drift which can occur otherwise. The mathematical fundamentals of this method may be found in [2].

\[
\text{if } \dot{\xi}_2 > \eta \text{ and } \dot{\xi}_2 > \eta \\
I_1 \omega_1 = M_m - C \cdot \theta - F_{s1} \cdot R \\
I_2 \omega_2 = C \cdot \theta - F_{s2} \cdot R
\]

\[
\text{if } \dot{\xi}_2 > \eta \text{ and } \dot{\xi}_2 \leq \eta \\
I_1 \omega_1 = M_m - C \cdot \theta - F_{s1} \cdot R \\
I_2 \omega_2 = C \cdot \theta - \min (C \cdot \theta, F_{s\text{Max}} \cdot R)
\]

\[
\text{if } \dot{\xi}_2 \leq \eta \text{ and } \dot{\xi}_2 \leq \eta \\
I_1 \omega_1 = M_m - C \cdot \theta - \min ((M_m - C \cdot \theta) + F_{s\text{Max}} \cdot R) \\
I_2 \omega_2 = C \cdot \theta - F_{s2} \cdot R
\]

Mode selection has to be done by means of an algorithm, designed for this purpose. The integration uses an event-driven method which detects phase switches and distributes the computation interval accordingly – in the present study this is achieved through Equations 5. On each subinterval, the system model is a set of ordinary non-stiff differential equations which can be computed by any standard ODE solver, requiring minimal, elementary programming.

4. Simulation and results
The aim of the simulations below is to present the possibility and benefits of the non-smooth approach to the study of the axle stick-slip torsional vibrations which may occur when motor vehicles start. Therefore, the model employed preserves only the most important forces and their relevant features.
The engine torque was kept constant. The model parameters are given in Table 1. Inertia and elasticity values have been computed using the model dimensions and material characteristics.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_1$, $I_2$</td>
<td>20</td>
<td>kg·m²</td>
</tr>
<tr>
<td>$C$</td>
<td>100</td>
<td>kN·m</td>
</tr>
<tr>
<td>$R$</td>
<td>0.2</td>
<td>m</td>
</tr>
<tr>
<td>$M_m$</td>
<td>38</td>
<td>N·m</td>
</tr>
<tr>
<td>$F_{s\text{Max}}$</td>
<td>100</td>
<td>N</td>
</tr>
<tr>
<td>$A$</td>
<td>20</td>
<td>s·m⁻¹</td>
</tr>
<tr>
<td>$B$</td>
<td>10</td>
<td>s·m⁻¹</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$&lt; 5 \cdot 10^{-3}$</td>
<td>m·s⁻¹</td>
</tr>
</tbody>
</table>

The complete description of the friction force, Equation 4, is accomplished by the saturated friction law. The expressions used in the present simulations are given in Equations 6 which derives from laws used in non-smooth simulations [27]. The rate of decrease of the saturated friction force with the creep is given by the dimensionless parameter $A$, while the rate of increase in the friction force for large creep values [33]. In the case of the setup model, the relative velocity in the contact point is given by $\dot{\xi} = \omega R$.

$$ F_s = F_{s\text{Max}} \left( \frac{1}{1 + A \dot{\xi}} + B \dot{\xi} \right) \tag{6} $$

Series of creep force values under different parameter values are calculated according to the estimated creepage. These creep force values are compared to the estimated creep force and a series of creep force residuals can be calculated. By searching the minimum value of these residuals, the friction coefficient can be identified. However, experimental data featured a large dispersion.

Hence, the values of friction parameters, $A$ and $B$, are only a rough approximation. The same problem had to be faced related to the saturation creep limit, $\eta$. However, in this latter case, the model proved to be insensitive to its values, as long as they are about an order of magnitude below the sliding velocities.

Nevertheless, experimental data made available the value of the setup stick-slip dominant frequency, i.e., around 16Hz.

The simulations performed are very accurate in reproducing the dominant frequency, Figure 4a. Stick phases may also be clearly distinguished in the phase-plane plot, Figure 4b.

Very similar solutions were obtained using a penalized linear expression for the unsaturated friction force, Equation 7.

$$ F_s = F_{s\text{Max}} \frac{\dot{\xi}}{\eta} \tag{7} $$

Integration time and the number of floating point operations is considerably higher in the case of the penalized model. More than that, this variant is becomes almost useless if $\eta$ becomes very small.

5 Conclusions

An event-driven integration of the railway vehicles driven axles is developed in the present paper. An experimental setup is measured to validate the model. Underlying mathematical theory is briefly reviewed and specific issues are solved.

The system is defined as a differential inclusion. The model proposed makes use of a non-smooth set-valued rolling friction law. The features of the law taken into account and the reduced DOF
allow integrating the system using a switch model. The model becomes a piecewise differential algebraic system and the number of DOF is variable during the integration.

The solution is remarkably similar to the one obtained with a penalized model but is definitely more efficient and stable from a numerical point of view. Results are in good agreement with experimental data.

References: