Some conservation laws for a porous medium equation through potential symmetries with free software

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Abstract: Potential symmetries, which are not local symmetries, were carried out for the porous medium equation \( u_t = (u^n)_{xx} + f(x)u^n u_x + g(x)u^m \) when it is written in a conserved form. These symmetries, by using free software Maxima, are realized as local symmetries of a related auxiliary system. In this paper we find the subclasses of weak and nonlinear self-adjoint porous medium systems. By using the property of nonlinear self-adjointness we construct some conservation laws associated with potential symmetries of the differential equation.

Key–Words: Symmetries, partial differential equation, self-adjointness, conservation laws

1 Introduction

The description of evolution processes where diffusion is combined with other effects, such as reaction, absorption and convection has attracted a lot of attention in the last decades.

Local symmetries admitted by a nonlinear partial differential equation (PDE) are useful to discover whether or not the equation can be linearized by an invertible mapping and construct an explicit linearization when one exists. A nonlinear scalar PDE is linearizable by an invertible contact (point) transformation if and only if it admits an infinite-parameter Lie group of contact transformations satisfying specific criteria [1012, 48]. An obvious limitation of group-theoretic methods based in local symmetries, in their utility for particular PDEs, is that many of these equations do not have local symmetries. It turns out that PDEs can admit nonlocal symmetries whose infinitesimal generators depend on integrals of the independent variables in some specific manner. It also happens that if a nonlinear scalar PDE does not admit an infinite-parameter Lie group of contact transformations, it is not linearizable by an invertible contact transformation. However, most of the interesting linearizations involve non-invertible transformations; such linearizations can be found by embedding given nonlinear PDEs in auxiliary systems of PDEs [10]. Krasilshchik and Vinograd [46, 47, 63] gave criteria which must be satisfied by nonlocal symmetries of a PDE when realized as local symmetries of a system of PDEs which covers the given PDE. Akhatov et al [2] gave nontrivial examples of nonlocal symmetries generated by heuristic procedures. By using nonlocal symmetries, some exact solutions which are not similarity solutions of (2) for special values of \( n \) were obtained by King [43, 44]. Nonlocal symmetries and Lie Backlund symmetries for this equation are well known [2, 35, 37, 51]. In [6] potential symmetries for the porous medium equation (1) were obtained. In order to find the potential symmetries of (1), equation (1) ought to be written a conserved form. When \( m = s + 1 \) and \( mg(x) = f'(x) \) equation (1) can be written in a conserved form

\[
 u_t = [(u^n)_x + f(x)u^m]_x 
\]

While more often that not the spatial-dependent factors are assumed to be constant, there is no fundamental reason to assume so. In fact, allowing for their spatial dependence enables one to incorporate additional factors into the study which may play an important role. For instance, in a porous medium this may account for intrinsic factors, like medium contamination with another material. Also in plasma, this may express the impact that solid impurities arising from the walls have on the enhancement of the radiation channel. The model equation to be considered here is the one-dimensional evolution equation involving diffusion, and convection

\[
 v_x = u \\
 v_t = [(u_x)^n]_x + \frac{f(x)}{m}u^m 
\] (1)

In a previous work [5], we study equation (1)
from the point of view of the theory of symmetry reductions in partial differential equations. We obtain the nonclassical symmetries admitted by (1), we list the different choices for functions \( f(x) \), and constants \( n \), and \( s \), for which equation (1) admits nonclassical reductions then, we use the transformations groups to reduce the equations to ordinary differential equations.

In a previous paper [2] we have derived the subclasses of equations which are self-adjoint.

Many equations having remarkable symmetry properties and physical significance are not self-adjoint. Therefore one cannot eliminate the nonlocal variables from conservation laws of these equations by setting \( v = u \), [12] generalized the concept of self-adjoint equations by introducing the definition of quasi-self-adjoint equations. A lot of attention has been paid on using the concept of self-adjointness to obtain conservation laws [9].

The aim of this paper is to determine, for system (1), the subclasses of systems which are nonlinear self-adjoint. We also determine, by using the notation and techniques of [12], some non-trivial conservation laws for (1) by using a potential generator.

2 Adjoint and self-adjoint nonlinear equations

The following definitions of adjoint equations and self-adjoint equations are applicable to any system of linear and non-linear differential equations, where the number of equations is equal to the number of dependent variables (see [11]), and contain the usual definitions for linear equations as a particular case. Since we will deal in our paper with scalar equations, we will formulate these definitions in the case of one dependent variable only.

Consider an \( s \)-th order partial differential equation

\[
F(x, u, u_1, \ldots, u_s) = 0
\]  

(2)

with independent variables \( x = (x^1, \ldots, x^n) \) and a dependent variable \( u \), where \( u_1 = \{u_i\}, \ u_2 = \{u_{ij}\}, \ldots \) denote the sets of the partial derivatives of the first, second, etc. orders, \( u_i = \partial u / \partial x^i, \ u_{ij} = \partial^2 u / \partial x^i \partial x^j \). The adjoint equation to (2) is

\[
F^*(x, u, v, u_1(1), v_1(1), \ldots, u_s, v_s) = 0,
\]  

(3)

with

\[
F^*(x, u, v, u_1(1), v_1(1), \ldots, u_s, v_s) = \frac{\delta(v F)}{\delta u},
\]  

(4)

where

\[
\delta u = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty} (-1)^s D_i \cdots D_s \frac{\partial}{\partial u_{i_1 \cdots i_s}}
\]  

(5)

denotes the variational derivatives (the Euler-Lagrange operator), and \( v \) is a new dependent variable. Here

\[
D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots
\]

are the total differentiations.

Eq. (2) is said to be self-adjoint if the equation obtained from the adjoint equation (3) by the substitution \( v = u \):

\[
F^*(x, u, u_1(1), v_1(1), \ldots, u_s, v_s) = 0,
\]

is identical with the original equation (2). In other words, if

\[
F^*(x, u, u_1(1), \ldots, u_s, u_s) = \phi(x, u, u_1(1), \ldots) F(x, u, u_1(1), \ldots, u_s).
\]  

(6)

2.1 General theorem on conservation laws

We use the following theorem on conservation laws proved in [12].

Theorem 1 Any Lie point, Lie-B"acklund or non-local symmetry

\[
X = \xi^i(x, u, u_1(1), \ldots) \frac{\partial}{\partial x^i} + \eta(x, u, u_1(1), \ldots) \frac{\partial}{\partial u}
\]  

(7)

defines a conservation law \( D_i(C^i) = 0 \) for the simultaneous system (2), (3). The conserved vector is given by

\[
C^i = \xi^i \mathcal{L} + W \left[ \frac{\partial \mathcal{L}}{\partial u_i} - D_j \left( \frac{\partial \mathcal{L}}{\partial u_{ij}} \right) \right] + D_j D_k \left[ \frac{\partial \mathcal{L}}{\partial u_{ijk}} \cdots \right] + \cdots 
\]  

(8)

where \( W \) and \( \mathcal{L} \) are defined as follows:

\[
W = \eta - \xi^i u_i, \quad \mathcal{L} = \varphi F(x, u, u_1(1), \ldots, u_s).
\]  

(9)
2.2 The class of self-adjoint equations

Let us single out systems of weak self-adjoint and nonlinear self-adjoint equations from the systems of the form (1). We introduce the formal Lagrangian for system (1)

\[ \mathcal{L} \equiv \bar{u}(v_x - u) + \bar{v}(u_t - (u^n)_x - f(x)u^m), \]  

(10)

where \( \bar{u} \) and \( \bar{v} \) are two new dependent variables.

The adjoint system for the system of differential equations (1) is defined by

\[
\begin{aligned}
F_1^* &\equiv \frac{\delta \mathcal{L}}{\delta u} = 0, \\
F_2^* &\equiv \frac{\delta \mathcal{L}}{\delta v} = 0,
\end{aligned}
\]

(11)

with

\[
\begin{aligned}
\frac{\delta \mathcal{L}}{\delta u} &\equiv \frac{\partial \mathcal{L}}{\partial u} - D_1 \frac{\partial \mathcal{L}}{\partial u_t} - D_2 \frac{\partial \mathcal{L}}{\partial u_x} + D_3 \frac{\partial \mathcal{L}}{\partial u_{xx}} + \cdots + D_x D_3 \frac{\partial \mathcal{L}}{\partial u_{xxx}}, \\
\frac{\delta \mathcal{L}}{\delta v} &\equiv \frac{\partial \mathcal{L}}{\partial v} - D_1 \frac{\partial \mathcal{L}}{\partial v_t} - D_2 \frac{\partial \mathcal{L}}{\partial v_x} + D_3 \frac{\partial \mathcal{L}}{\partial v_{xx}} + \cdots + D_x D_3 \frac{\partial \mathcal{L}}{\partial v_{xxx}},
\end{aligned}
\]

(12)

where \( D_1 \) and \( D_2 \) denote the total differentiations with respect to \( t \) and \( x \) respectively.

Taking into account the Eqs. (10)-(11) the adjoint system for system (1) is

\[
\begin{aligned}
F_1^* &\equiv n u^{n-1} v_x - f u^{m-1} \bar{v} - \bar{u} = 0, \\
F_2^* &\equiv -\bar{v}_t - \bar{u}_x = 0.
\end{aligned}
\]

(14)

Setting

\[ \bar{u} = \phi(t,x,u,v), \quad \bar{v} = \psi(t,x,u,v). \]

(15)

\[
\begin{aligned}
F_1^* &\equiv \psi_v n u^{n-1} v_x + \psi_u n u^{n-1} u_x + \psi_x n u^{n-1} - f \psi u^{m-1} - \phi = 0, \\
F_2^* &\equiv -\phi_x v_x - \phi_v u_t - \phi_u u_x - \phi_t = 0.
\end{aligned}
\]

(16)

The nonlinear differential system (1) is said to be nonlinear self-adjoint if each equations \( F_i^* \) of the adjoint system (16) coincides with \( \lambda_i F_1 + \mu_i F_2 \) after substitution (17) \[ ? \]:

\[ \bar{u} = \phi(t,x,u,v), \quad \bar{v} = \psi(t,x,u,v). \]

(17)

In other words system (1) is said to be nonlinear self-adjoint if the adjoint system (16) obeys the condition

\[
\begin{aligned}
F_1^* |_{\bar{u}=\phi(t,x,u,v), \bar{v}=\psi(t,x,u,v)} &= F_1 \lambda_1 + F_2 \mu_1, \\
F_2^* |_{\bar{u}=\phi(t,x,u,v), \bar{v}=\psi(t,x,u,v)} &= F_1 \lambda_2 + F_2 \mu_2
\end{aligned}
\]

(18)

with regular undetermined coefficients \( \lambda_i, \mu_i (i = 1, 2) \).

When functions \( \phi(t,x,u,v) \) and \( \psi(t,x,u,v) \) are such that \( \phi_x \neq 0 \) or \( \phi_t \neq 0 \) and \( \phi_u \neq 0 \) or \( \psi_x \neq 0 \) or \( \psi_t \neq 0 \) and \( \psi_v \neq 0 \) the system is called weak-self-adjoint.

For system(1) condition (18) becomes

\[
\begin{aligned}
\psi_v n u^{n-1} v_x - \lambda_1 v_x - \mu_1 v_t + \mu_1 n u^{n-1} u_x + \psi_x n u^{n-1} + f \mu u^m - f \psi u^{m-1} + \lambda_1 u - \phi &= 0, \\
-\lambda_2 v_x - \phi_v v_x - \mu_2 v_t - \psi_v v_t + \mu_2 n u^{n-1} u_x - \phi_u u_x - \psi_u u_t + \frac{f \mu u^m}{m} + \lambda_2 u - \phi_x - \psi_t &= 0.
\end{aligned}
\]

(19)

Using the differential consequences of (17), since \( \phi \) and \( \psi \) do not depend on the derivatives \( u_t, v_t, u_{xx}, \ldots \) equation (18) split into the following system

\[
\begin{aligned}
\mu_1 &= 0, \quad \mu_2 = -\psi_v, \\
\psi_v n u^n + \psi_x n u^{n-1} - f \psi u^{m-1} - \phi &= 0, \\
-\psi_v n u^n - \phi_u u_x &= 0, \\
-\frac{f \psi_u u^m}{m} - \phi_v u - \phi_x - \psi_t &= 0.
\end{aligned}
\]

(20)

From (20) we get

\[ \phi = \eta - \psi_v u^n \]

where \( \eta = \eta(t,x,v) \) and substituting into the remaining equation (20) we get

\[
\begin{aligned}
\psi_v n u^n + \psi_x n u^{n-1} - f \psi u^{m-1} - \eta &= 0, \\
\psi_v n u^n + \psi_x n u^{n-1} - f \psi u^{m-1} - \eta &= 0.
\end{aligned}
\]

We can distinguish the following cases:
Theorem 2 System (1) for \( n = -1 \) \( m = -1 \) and \( f(x) \) arbitrary is weak and nonlinear self-adjoint with
\[
\psi(x, t, v) = \beta(v, t) e^{-\int f(x) dx} \\
\phi(x, t, u, v) = -\frac{\beta_v e^{-\int f(x) dx}}{u}.
\]
(21)
such that \( \beta_t - \beta_v v = 0 \).

Theorem 3 System (1) for \( n = -1 \) \( m = 1 \) \( f(x) = cx \) is not weak self-adjoint however it is and nonlinear self-adjoint with
\[
\psi(t) = k e^{ct} \\
\phi(x, t) = -ke^{ct} x.
\]
(22)

In order to apply theorem 1 to the nonlinear self-adjoint system (1), we will write generators of point transformation group admitted by (1) in the form
\[
X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v}.
\]
The conservation law will be written
\[
D_t(C^1) + D_x(C^2) = 0.
\]
(23)

1- Let us apply theorem 1 to the nonlinear self-adjoint system (1) with \( n = -1 \), \( m = -1 \) \( f = f(x) \) arbitrary, with
\[
\psi(x, t, v) = \beta(v, t) e^{-\int f(x) dx} \\
\phi(x, t, u, v) = -\frac{\beta_v e^{-\int f(x) dx}}{u}.
\]
(24)
such that \( \beta_t - \beta_v v = 0 \).

In this case we compute the conserved vectors by (8), provided by the following potential symmetry of (1):
\[
X = \alpha(v, t) e^{\int f(x) dx} \frac{\partial}{\partial x} \\
-\alpha_v u^2 + f \alpha u e^{\int f(x) dx} \frac{\partial}{\partial u}.
\]
(25)
where
\[
\alpha_t + \alpha v v = 0.
\]
In this case we have
\[
W_1 = -(\alpha_v u^2 + f \alpha u) e^{\int f(x) dx} \\
-\alpha(v, t) e^{\int f(x) dx} u_x.
\]
(26)
and Eqs. (8) yield the conservation law (23) where
\[
C^1 = -\alpha \beta u \\
C^2 = -\frac{\alpha \beta u_x}{u^2} - \frac{\alpha \beta f}{u} + \alpha \beta_v - \alpha_v \beta
\]
with
\[
\alpha_t + \alpha_v v = 0
\]
and where
\[
\beta_t - \beta_v v = 0
\]

2- Let us apply theorem 1 to the nonlinear self-adjoint system (1) where \( n = -1 \), \( m = 1 \) and \( f = cx \) with
\[
\psi(t) = k e^{ct} \\
\phi(x, t) = -ke^{ct} x.
\]
(28)

In this case we compute the conserved vectors by (8), provided by the following potential symmetry of (1):
\[
X = \frac{x v}{2} \frac{\partial}{\partial x} - \frac{(v u + u^2 x)}{2} \frac{\partial}{\partial u} + \frac{1}{2} \frac{\partial}{\partial v}.
\]
(29)

In this case we have
\[
W_1 = -(\frac{v u}{2} + \frac{u^2 x}{2}) - \frac{x v}{2} u_x
\]
(30)
and Eqs. (8) yield the conservation law (23) where
\[
v = \int u dx
\]
(31)

3 Conclusions

The concepts of self-adjoint and quasi self-adjoint equations were introduced by NH Ibragimov in [10, 13]. In [7, 14] the concept of self-adjoint and quasi self-adjoint equations have been generalized by introducing the definition of weak self-adjoint equations and nonlinear self-adjoint equation. In this paper we found some classes of nonlinear self adjoint porous medium systems. By using the property of nonlinear self-adjointness of a porous medium system and the general theorem of conservation laws [12] we have constructed some nontrivial conservation laws for this porous medium system associated with potential generators of the differential equations.
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