Abstract: Large amplitude (geometrically non-linear) vibrations of doubly curved shallow shells with rectangular base under the low-velocity impact by an elastic sphere are investigated. It is assumed that the shell is simply supported and partial differential equations are obtained in terms of shell’s transverse displacement and Airy’s stress function. The local bearing of the shell and impactor’s materials is neglected with respect to the shell deflection in the contact region. The equations of motion are reduced to a set of infinite nonlinear ordinary differential equations of the second order in time and with cubic and quadratic nonlinearities in terms of the generalized displacements. The contact force is considered to be a small value. Assuming that only two natural modes of vibrations dominate during the process of impact and applying the method of multiple time scales, the set of equations is obtained, which allows one to find the time dependence of the contact force and to determine the contact duration and the maximal contact force.

Key–Words: Doubly curved shallow shell rectangular in base, Impact interaction, Method of multiple time scales, Impact induced resonance

1 Introduction

Doubly curved panels are widely used in aeronautics, aerospace and civil engineering and are subjected to dynamic loads that can cause vibration amplitude of the order of the shell thickness, giving rise to significant non-linear phenomena [1]–[4].

A review of the literature devoted to dynamic behaviour of curved panels and shells could be found in Amabili and Paidoussis [5], as well as in [3], wherein it has been emphasized that free vibrations of doubly curved shallow shells were studied in the majority of papers either utilizing a slightly modified version of the Donnell’s theory taking into account the double curvature [1, 6] or the nonlinear first-order theory of shells [7, 8].

Large-amplitude vibrations of doubly curved shallow shells with rectangular base, simply supported at the four edges and subjected to harmonic excitation were investigated in [3], while chaotic vibrations were analyzed in [4]. It has been revealed that such an important nonlinear phenomenon as the occurrence of internal resonances in the problems considered in [3] and [4] is of fundamental importance in the study of curved shells.

In spite of the fact that the impact theory is substantially developed, there is a limited number of papers devoted to the problem of impact over geometrically nonlinear shells.

The nonlinear impact response of laminated composite cylindrical and doubly curved shells was analyzed using a modified Hertzian contact law in [9] via a finite element model, which was developed based on Sander’s shell theory involving shear deformation effects and nonlinearity due to large deflection. A nine-noded isoparametric quadrilateral element was used to model the curved shell. The nonlinear time dependent equations were solved using an iterative scheme and Newmark’s method. Numerical results for the contact force and center deflection histories were presented for various impactor conditions, shell geometry and boundary conditions.

Later large deflection dynamic responses of laminated composite cylindrical shells under impact have been analyzed in [10] by the geometrically nonlinear finite element method based on a generalized Sander’s shell theory with the first order shear deformation and the von Karman large deflection assumption.

Nonlinear dynamic response for shallow spherical moderate thick shells with damage under low velocity impact has been studied in [11] by using the or-
thogonal collocation point method and the Newmark method to discrete the unknown variable function in space and in time domain, respectively, and the whole problem is solved by the iterative method. Further this approach was generalized for investigating dynamic response of elasto-plastic laminated composite shallow spherical shell under low velocity impact [12] and nonlinear dynamic response for functionally graded shallow spherical shell under low velocity impact in thermal environment [13].

The nonlinear transient response of laminated composite shell panels subjected to low velocity impact in hygrothermal environments was investigated in [14] using finite element method considering doubly curved thick shells involving large deformations with Green-Lagrange strains. The analysis was carried out using quadratic eight-noded isoparametric element. A modified Hertzian contact law was incorporated into the finite element program to evaluate the impact force. The nonlinear equation was solved using the Newmark average acceleration method in conjunction with an incremental modified Newton-Raphson scheme. A parametric study was carried out to investigate the effects of the curvature and side to thickness ratios of simply supported composite cylindrical and spherical shell panels.

The impact behaviour and the impact-induced damage in laminated composite cylindrical shell subjected to transverse impact by a foreign object were studied in [15] using three-dimensional non-linear transient dynamic finite element formulation. Non-linear system of equations resulting from non-linear strain displacement relation and non-linear contact loading was solved using Newton-Raphson incremental-iterative method. Some example problems of graphite/epoxy cylindrical shell panels were considered with variation of impactor and laminate parameters and influence of geometrical non-linear effect on the impact response and the resulting damage was investigated.

In the present paper, a new approach is proposed for the analysis of the impact interactions of nonlinear doubly curved shallow shells with rectangular base under the low-velocity impact by an elastic sphere. It is assumed that the shell is simply supported and partial differential equations are obtained in terms of shell’s transverse displacement and Airy’s stress function. The local bearing of the shell and impactor’s materials is neglected with respect to the shell deflection in the contact region. The equations of motion are reduced to a set of infinite nonlinear ordinary differential equations of the second order in time and with cubic and quadratic nonlinearities in terms of the generalized displacements. The contact force is considered to be a small value. Assuming that only two natural modes of vibrations dominate during the process of impact and applying the method of multiple time scales, the set of equations is obtained, which allows one to investigate different possible types of internal resonance, to find the time dependence of the contact force and to determine the contact duration and the maximal contact force.

2 Problem formulation and governing equations

Assume that an elastic or rigid sphere of mass $M$ moves along the $z$-axis towards a thin walled doubly curved shell with thickness $h$, curvilinear lengths $a$ and $b$, principle curvatures $k_x$ and $k_y$ and rectangular base, as shown in Fig. 1. Impact occurs at the moment $t = 0$ with the velocity $\varepsilon V_0$ at the point $N$ with Cartesian coordinates $x_0, y_0$.

According to Donnell’s nonlinear shallow shell theory, the equations of motion could be obtained in terms of lateral deflection $w$ and Airy’s stress function $\phi$ [16]

$$
\frac{D}{h} \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 \phi}{\partial x^2} - 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 \phi}{\partial x \partial y} + k_y \frac{\partial^2 \phi}{\partial x^2} + k_x \frac{\partial^2 \phi}{\partial y^2} + \varepsilon \frac{F}{h} \frac{1}{\rho} \ddot{w},
$$

Figure 1: Geometry of the doubly curved shallow shell
where \( D = \frac{Eh^3}{12(1-\nu^2)} \) is the cylindrical rigidity, \( \rho \) is the density, \( E \) and \( \nu \) are the elastic modulus and Poisson’s ratio, respectively, \( t \) is time, \( F = P(t)\delta(x-x_0)\delta(y-y_0) \) is the contact force, \( P(t) \) is yet unknown function, \( \delta \) is the Dirac delta function, \( x \) and \( y \) are Cartesian coordinates, overdots denote time-derivatives, \( \phi(x,y) \) is the stress function which is the potential of the in-plane force resultants

\[
N_x = E \frac{\partial^2 \phi}{\partial y^2}, \quad N_y = E \frac{\partial^2 \phi}{\partial x^2}, \quad N_{xy} = -h \frac{\partial^2 \phi}{\partial x \partial y}.
\]

(3)

The equation of motion of the sphere is written as

\[
M \ddot{z} = -P(t)
\]

(4)

subjected to the initial conditions

\[
z(0) = 0, \quad \dot{z}(0) = \varepsilon V_0,
\]

(5)

where \( z(t) \) is the displacement of the sphere, in so doing

\[
z(t) = w(x_0,y_0,t).
\]

(6)

Considering a simply supported shell with movable edges, the following conditions should be imposed at each edge:

at \( x = 0, \ a \)

\[
w = 0, \quad \int_0^b N_{xy} dy = 0, \quad N_x = 0, \quad M_x = 0,
\]

(7)

and at \( y = 0, \ b \)

\[
w = 0, \quad \int_0^a N_{xy} dx = 0, \quad N_y = 0, \quad M_y = 0,
\]

(8)

where \( M_x \) and \( M_y \) are the moment resultants.

The suitable trial function that satisfies the geometric boundary conditions is

\[
w(x,y,t) = \sum_{p=1}^{\tilde{p}} \sum_{q=1}^{\tilde{q}} \xi_{pq}(t) \sin \left( \frac{p \pi x}{a} \right) \sin \left( \frac{q \pi y}{b} \right),
\]

(9)

where \( p \) and \( q \) are the number of half-waves in \( x \) and \( y \) directions, respectively, and \( \xi_{pq}(t) \) are the generalized coordinates. Moreover, \( \tilde{p} \) and \( \tilde{q} \) are integers indicating the number of terms in the expansion.

Substituting (9) in (6) and using (4), we obtain

\[
P(t) = -M \sum_{p=1}^{\tilde{p}} \sum_{q=1}^{\tilde{q}} \xi_{pq}(t) \sin \left( \frac{p \pi x_0}{a} \right) \sin \left( \frac{q \pi y_0}{b} \right),
\]

(10)

In order to find the solution of the set of equations (1) and (2), it is necessary first to obtain the solution of Eq. (2). For this purpose, let us substitute (9) in the right-hand side of Eq. (2) and seek the solution of the equation obtained in the form

\[
\phi(x,y,t) = \sum_{m=1}^{\tilde{m}} \sum_{n=1}^{\tilde{n}} A_{mn}(t) \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right),
\]

(11)

where \( A_{mn}(t) \) are yet unknown functions.

Substituting (9) and (11) in Eq. (2) and using the orthogonality conditions of sines within the segments \( 0 \leq x \leq a \) and \( 0 \leq y \leq b \), we have

\[
A_{mn}(t) = \frac{E}{\pi^2} K_{mn} \xi_{mn}(t) + \frac{4E}{a^4 b^2} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^{-2} \sum_{k} \sum_{l} \sum_{p} \sum_{q} B_{pqklmn} \xi_{pq}(t) \xi_{kl}(t),
\]

(12)

where

\[
B_{pqklmn} = pql \left( B_{pqklmn}^{(1)} - p^2 l^2 B_{pqklmn}^{(2)} \right),
\]

\[
B_{pqklmn}^{(1)} = \int_0^a \int_0^b \sin \left( \frac{p \pi x}{a} \right) \sin \left( \frac{q \pi y}{b} \right) \sin \left( \frac{k \pi x}{a} \right) \times \sin \left( \frac{l \pi y}{b} \right) \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) dx dy,
\]

\[
B_{pqklmn}^{(2)} = \int_0^a \int_0^b \cos \left( \frac{p \pi x}{a} \right) \cos \left( \frac{q \pi y}{b} \right) \cos \left( \frac{k \pi x}{a} \right) \times \cos \left( \frac{l \pi y}{b} \right) \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) dx dy,
\]

\[
K_{mn} = \left( k_y \frac{m^2}{a^2} + k_x \frac{n^2}{b^2} \right)^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^{-2}.
\]

Substituting then (9)–(12) in Eq. (1) and using the orthogonality condition of sines within the segments \( 0 \leq x \leq a \) and \( 0 \leq y \leq b \), we obtain an infinite set of coupled nonlinear ordinary differential equations of the second order in time for defining the generalized coordinates

\[
\ddot{\xi}_{mn}(t) + \Omega_{mn}^2 \xi_{mn}(t) + \frac{8\pi^2 E}{a^4 b^2 \rho} \sum_{p} \sum_{q} \sum_{k} \sum_{l} B_{pqklmn} \left( K_{kl} - \frac{1}{2} K_{mn} \right)
\]

(13)
where
\[ \omega_{mm}^{2} = \frac{E}{\rho} \left[ \frac{\pi^{4} h_{1}^{2}}{12(1 - \nu^{2})} \left( \frac{m^{2}}{a^{2}} + \frac{n^{2}}{b^{2}} \right)^{2} + K_{nn} \right]. \]

### 3 Method of solution

Suppose that only two natural modes of vibrations are excited during the process of impact, namely, \( \Omega_{\alpha\beta} \) and \( \Omega_{\gamma\delta} \), and apply the method of multiple time scales [17] for constructing the solution of Eqs. (13)

\[ \xi_{ij}(t) = \varepsilon X_{ij}^{1}(T_{0}, T_{1}) + \varepsilon^{2} X_{ij}^{2}(T_{0}, T_{1}), \quad (14) \]

where \( i,j = \alpha\beta \) or \( \gamma\delta \), and \( T_{n} = \varepsilon^{n} t \) are new independent variables, among them: \( T_{0} = t \) is a fast scale characterizing motions with the natural frequencies, and \( T_{1} = \varepsilon t \) is a slow scale characterizing the modulation of the amplitudes and phases of the modes with nonlinearity.

Considering that
\[ \frac{d^{2}}{dt^{2}} \xi_{ij} = \varepsilon \left( D_{0}^{2} X_{ij}^{1} \right) + \varepsilon^{2} \left( D_{0}^{2} X_{ij}^{2} + 2D_{0} D_{1} X_{ij}^{1} \right), \]

where \( D_{n}^{i} = \partial^{n}/\partial T_{n}^{i} \) \((n = 1, 2; \ i = 0, 1)\), and substituting the proposed solution (14) in Eq. (13), after equating the coefficients at like powers of \( \varepsilon \) to zero, we are led to a set of recurrence equations to various orders:

- to order \( \varepsilon \)
  \[ D_{0}^{2} X_{1}^{1} + \Omega_{1}^{2} X_{1}^{1} = 0, \quad (15) \]
  \[ D_{0}^{2} X_{2}^{1} + \Omega_{2}^{2} X_{2}^{1} = 0; \quad (16) \]

- to order \( \varepsilon^{2} \)
  \[ D_{0}^{2} X_{1}^{2} + \Omega_{1}^{2} X_{1}^{2} = -2D_{0} D_{1} X_{1}^{1}, \quad \]
  \[ - p_{11} \left( X_{1}^{1} \right)^{2} - p_{22} \left( X_{1}^{2} \right)^{2}, \quad \]
  \[ - (p_{12} + p_{21}) X_{1}^{1} X_{1}^{2}, \quad \]
  \[ - d_{1} D_{0}^{2} X_{1}^{1} - d_{2} D_{0}^{2} X_{1}^{2}, \quad (17) \]

We shall seek the solution of Eqs. (15) and (16) in the form

\[ X_{1}^{1} = A_{1}(T_{1}) \exp(i\Omega_{1} T_{0}) + A_{1}^{*}(T_{1}) \exp(-i\Omega_{1} T_{0}), \quad (19) \]

\[ X_{1}^{2} = A_{2}(T_{1}) \exp(i\Omega_{2} T_{0}) + A_{2}^{*}(T_{1}) \exp(-i\Omega_{2} T_{0}), \quad (20) \]

where \( A_{1}(T_{1}) \) and \( A_{2}(T_{1}) \) are unknown complex functions, and \( A_{1}(T_{1}) \) and \( A_{2}(T_{1}) \) are their complex conjugates.
Substituting (19) and (20) in the right-hand side of Eqs. (17) and (18) yields

\[
\begin{align*}
D_0^2 X_1^2 &+ \Omega_i^2 X_1^2 = -2i\Omega_1 D_1 A_1 \exp(i\Omega_1 T_0) \\
&- p_{11} \left[ A_1^2 \exp(2i\Omega_1 T_0) + A_1 \tilde{A}_1 \right] \\
&- p_{22} \left[ A_2^2 \exp(2i\Omega_2 T_0) + A_2 \tilde{A}_2 \right] \\
&- (p_{12} + p_{21}) \{ A_1 A_2 \exp[i(\Omega_1 + \Omega_2) T_0] \\
&+ A_1 \tilde{A}_2 \exp[i(\Omega_1 - \Omega_2) T_0] \} \\
&+ d_1 \Omega_i^2 A_1 \exp(i\Omega_1 T_0) \\
&+ d_2 \Omega_i^2 A_2 \exp(i\Omega_2 T_0) + cc, \quad (21)
\end{align*}
\]

\[
\begin{align*}
D_0^2 X_2^2 &+ \Omega_i^2 X_2^2 = -2i\Omega_2 D_1 A_2 \exp(i\Omega_2 T_0) \\
&- q_{11} \left[ A_1^2 \exp(2i\Omega_1 T_0) + A_1 \tilde{A}_1 \right] \\
&- q_{22} \left[ A_2^2 \exp(2i\Omega_2 T_0) + A_2 \tilde{A}_2 \right] \\
&- (q_{12} + q_{21}) \{ A_1 A_2 \exp[i(\Omega_1 + \Omega_2) T_0] \\
&+ A_1 \tilde{A}_2 \exp[i(\Omega_1 - \Omega_2) T_0] \} \\
&+ b_1 \Omega_i^2 A_1 \exp(i\Omega_1 T_0) \\
&+ b_2 \Omega_i^2 A_2 \exp(i\Omega_2 T_0) + cc, \quad (22)
\end{align*}
\]

where \(cc\) is the complex conjugate part to the preceding terms.

Reference to Eqs. (21) and (22) shows that the following types of the two-to-one internal resonance, when one natural frequency is twice the other natural frequency, could occur:

\[
\begin{align*}
\Omega_1 &= 2\Omega_2, \quad (23) \\
\Omega_2 &= 2\Omega_1. \quad (24)
\end{align*}
\]

### 4 Two-to-one internal resonance

Suppose that \(\Omega_1 \approx 2\Omega_2\). Then equating to zero in Eqs. (21) and (22) the terms producing the secular terms, we obtain the following solvability conditions:

\[
\begin{align*}
2i\Omega_1 D_1 A_1 + p_{22} A_2^2 - d_1 \Omega_i^2 A_1 &= 0, \quad (25) \\
2i\Omega_2 D_1 A_2 + (q_{21} + q_{12}) A_1 \tilde{A}_2 - b_2 \Omega_i^2 A_2 &= 0. \quad (26)
\end{align*}
\]

The last terms in Eqs. (25) and (26) describe the influence of the impactor and the coordinates of the impact point on the two-to-one internal resonance via the coefficients \(d_1\) and \(b_2\).

Multiplying Eqs. (25) and (26) by \(\tilde{A}_1\) and \(\tilde{A}_2\), respectively, and writing their complex conjugate equations yields

\[
\begin{align*}
2i\Omega_1 \tilde{A}_1 D_1 A_1 + p_{22} \tilde{A}_1 A_2^2 - d_1 \Omega_i^2 \tilde{A}_1 A_1 &= 0, \quad (27) \\
-2i\Omega_1 A_1 D_1 \tilde{A}_1 + p_{22} A_1 \tilde{A}_2^2 - d_1 \Omega_i^2 A_1 \tilde{A}_1 &= 0, \quad (28)
\end{align*}
\]

\[
2i\Omega_2 \tilde{A}_2 D_1 A_2 + (q_{21} + q_{12}) A_1 \tilde{A}_2 - b_2 \Omega_i^2 \tilde{A}_2 A_2 &= 0, \quad (29)
\]

\[
-2i\Omega_2 A_2 D_1 \tilde{A}_2 + (q_{21} + q_{12}) \tilde{A}_1 A_2^2 - b_2 \Omega_i^2 \tilde{A}_2 A_2 &= 0. \quad (30)
\]

Adding every pair of the mutually adjoint equations with each other and subtracting one from another, and considering that \(A_1\) and \(A_2\) could be represented in the polar form

\[
A_i = a_i \exp(i\varphi_i) \quad (i = 1, 2), \quad (31)
\]

as a result we have

\[
\begin{align*}
\left( a_1^2 \right) &= -\frac{p_{22}}{\Omega_1} a_1 a_2^2 \sin \delta, \quad (32) \\
\dot{\varphi}_1 &= \frac{1}{2} \sigma_1 - \frac{1}{2} \frac{p_{22}}{\Omega_1} a_1^{-1} a_2^2 \cos \delta = 0, \quad (33) \\
\left( a_2^2 \right) &= \frac{q_{21} + q_{12}}{\Omega_2} a_1 a_2^2 \sin \delta, \quad (34) \\
\dot{\varphi}_2 &= \frac{1}{2} \sigma_2 - \frac{1}{2} \frac{q_{21} + q_{12}}{\Omega_2} a_1 \cos \delta = 0, \quad (35)
\end{align*}
\]

where \(\delta = 2\varphi_2 - \varphi_1, \sigma_1 = d_1 \Omega_1, \sigma_2 = b_2 \Omega_2\), and an overdot denotes the derivative with respect to \(T_1\).

Using Eqs. (32) and (34), it is possible to obtain the first integral of the set of Eqs. (32)-(35), which is the law of energy conservation. Really, multiply Eq. (32) by \(\Omega_1 p_{22}\) and Eq. (34) by \(\Omega_2 (q_{21} + q_{12})^{-1}\), respectively, and add the equations obtained. As a result we have

\[
\left( \frac{q_{21} + q_{12}}{\Omega_2} \right) \left( a_1^2 \right) + \frac{p_{22}}{\Omega_1} \left( a_2^2 \right) = 0. \quad (36)
\]

Multiplying (36) by \(MV_0\) and integrating over \(T_1\), we are led to the law of energy conservation

\[
MV_0 \left( \frac{q_{21} + q_{12}}{\Omega_2} a_1^2 + \frac{p_{22}}{\Omega_1} a_2^2 \right) = T_0. \quad (37)
\]

where \(T_0\) is the initial energy.

Considering that \(T_0 = \frac{1}{2} MV_0^2\), Eq. (37) is reduced to the following form:

\[
\frac{q_{21} + q_{12}}{\Omega_2} a_1^2 + \frac{p_{22}}{\Omega_1} a_2^2 = \frac{V_0}{2}. \quad (38)
\]

In order to satisfy Eq. (37), let us introduce into consideration a new function \(\xi(T_1)\) in the following form:

\[
a_1^2 = \frac{\Omega_2}{q_{21} + q_{12}} E_0 \xi(T_1), \quad a_2^2 = \frac{\Omega_1}{p_{22}} E_0 \left[ 1 - \xi(T_1) \right], \quad (39)
\]

where \(E_0 = V_0/2\).
It is easy to verify by the direct substitution that formulas (39) satisfy Eq. (37), while the value \( \xi(0) \) (0 \( \leq \xi(0) \leq 1 \)) governs the energy distribution between two subsystems, \( X_1 \) and \( X_2 \), at the moment of impact.

Substituting (39) in Eq. (32) yields
\[
\dot{\xi} = -b\sqrt{\xi(1-\xi)} \sin \delta, \tag{40}
\]
where
\[
b = \sqrt{E_0} \sqrt{\frac{q_{21} + q_{12}}{\Omega_2}}.
\]

Subtracting Eq. (33) from the doubled Eq. (35), we have
\[
\dot{\delta} = -b \frac{1-3\xi}{2\sqrt{\xi}} \cos \delta - \Sigma, \tag{41}
\]
where \( \Sigma = \sigma_2 - \frac{1}{2}\sigma_1 = \frac{M\Omega_2}{\rho ab} \left( s_2^2 - s_1^2 \right). \)

Equation (41) could be rewritten in another form considering that
\[
\dot{\delta} = \frac{d\delta}{d\xi} \dot{\xi},
\]
or with due account for (40)
\[
\dot{\delta} = -\frac{d\delta}{d\xi} b \sqrt{\xi}(1-\xi) \sin \delta. \tag{42}
\]

Substituting (42) in Eq. (41) yields
\[
\sqrt{\xi}(1-\xi) \frac{d \cos \delta}{d\xi} + \frac{1-3\xi}{2\sqrt{\xi}} \cos \delta + \frac{\Sigma}{b} = 0. \tag{43}
\]

Integrating (43), we have
\[
\cos \delta = \frac{G_0}{\sqrt{\xi}(1-\xi)} \left( \frac{\Sigma \sqrt{\xi}}{b(1-\xi)} \right), \tag{44}
\]
where \( G_0 \) is a constant of integration to be determined from the initial conditions.

Based on relationship (44), it is possible to introduce into consideration the stream function \( G(\delta, \xi) \) of the phase fluid on the plane \( \delta, \xi \) such that
\[
G(\delta, \xi) = \sqrt{\xi}(1-\xi) \cos \delta + e\xi = G_0, \tag{45}
\]
which is one more first integral of the set of Eqs. (32)-(35), and \( e = \Sigma b^{-1}. \)

It is easy to verify that the function (45) is really a stream function, since
\[
v_\delta = \dot{\delta} = -b \frac{\partial G}{\partial \xi}, \quad v_\xi = \dot{\xi} = b \frac{\partial G}{\partial \delta}. \tag{46}
\]

In order to find the \( T_1 \)-dependence of \( \xi \), it is necessary to express \( \sin \delta \) in terms of \( \xi \) in Eqs. (40) with a help of relationship (44). As a result we obtain
\[
\dot{\xi} = -b\sqrt{\xi(1-\xi)^2 - (G_0 - e\xi)^2},
\]
or
\[
\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi^3 - (2+\varepsilon^2)\xi^2 + (1+2eG_0)\xi - G_0^2}} = -bT_1, \tag{47}
\]
where \( \xi_0 \) is the initial magnitude of the function \( \xi = \xi(T_1) \). The value \( \xi_0 (\xi_0 \leq 1) \) governs the energy distribution between two subsystems, \( X_1 \) and \( X_2 \), at the moment of impact.

In other words, the calculation of the \( T_1 \)-dependence of \( \xi \) is reduced to the calculation of the incomplete elliptic integral in the left hand-side of (47).

### 4.1 Initial conditions

In order to construct the final solution of the problem under consideration, i.e. to solve the set of Eqs. (32)-(35) involving the functions \( a_1(T_1), a_2(T_1), \) or \( \xi(T_1) \), as well as \( \varphi_1(T_1), \) and \( \varphi_2(T_1), \) or \( \delta(T_1) \), it is necessary to use the initial conditions
\[
w(x, y, 0) = 0, \tag{48}
\]
\[
\dot{w}(x_0, y_0, 0) = \varepsilon V_0, \tag{49}
\]
\[
\frac{q_{21} + q_{12}}{\Omega_2} a_1^2(0) + \frac{p_{22}}{\Omega_1} a_2^2(0) = \frac{V_0}{2}. \tag{50}
\]

The two-term relationship for the displacement \( w(9) \) within an accuracy of \( \varepsilon \) according to (14) has the form
\[
w(x, y, t) = \varepsilon \left[ X_{a \delta}(T_0, T_1) \sin \left( \frac{\alpha \pi}{a} \right) \sin \left( \frac{\delta \pi}{b} \right) \right] + O(\varepsilon^2). \tag{51}
\]

Substituting (19) and (20) in (51) with due account for (32) yields
\[
w(x, y, t) = 2\varepsilon \left\{ a_1(\varepsilon t) \cos [\omega_1 t + \varphi_1(\varepsilon t)] \
\times \sin \left( \frac{\alpha \pi}{a} \right) \sin \left( \frac{\delta \pi}{b} \right) \right. \\
+ \omega_2(\varepsilon t) \cos [\omega_2 t + \varphi_2(\varepsilon t)] \sin \left( \frac{\gamma \pi}{a} \right) \sin \left( \frac{\delta \pi}{b} \right) \left. \right\} \\
+ O(\varepsilon^2). \tag{52}
\]

Differentiating (52) with respect to time \( t \) and limiting ourselves by the terms of the order of \( \varepsilon \), we could find the velocity of the shell at the point of impact as follows
\[
\dot{w}(x_0, y_0, t) = -2\varepsilon \left\{ \omega_1 a_1(\varepsilon t) \sin [\omega_1 t + \varphi_1(\varepsilon t)] \\
+ \omega_2 a_2(\varepsilon t) \sin [\omega_2 t + \varphi_2(\varepsilon t)] \right\} + O(\varepsilon^2). \tag{53}
\]
Substituting (52) in the first initial condition (48) and assuming that \( a_1(0) > 0 \) and \( a_2(0) > 0 \), we have
\[
\cos \varphi_1(0) = 0, \quad \cos \varphi_2(0) = 0, \tag{54}
\]
whence it follows that
\[
\varphi_1(0) = \pm \frac{\pi}{2}, \quad \varphi_2(0) = \pm \frac{\pi}{2}, \tag{55}
\]
i.e.,
\[
\cos \delta_0 = \cos [2\varphi_2(0) - \varphi_1(0)] = 0, \tag{56}
\]
and
\[
\delta_0 = \pm \frac{\pi}{2} \pm 2\pi n. \tag{57}
\]
The signs in (55) should be chosen considering the fact that the initial amplitudes are positive values, i.e. \( a_1(0) > 0 \) and \( a_2(0) > 0 \). Assume for definiteness that
\[
\varphi_1(0) = -\frac{\pi}{2}, \quad \varphi_2(0) = \frac{\pi}{2}. \tag{58}
\]
Substituting now (53) in the second initial condition (49) with due account for (58), we obtain
\[
-\omega_1 s_1 a_1(0) + \omega_2 s_2 a_2(0) = -\frac{V_0}{2}. \tag{59}
\]
From equations (50) and (59) we could determine the initial amplitudes
\[
a_2(0) = \frac{\omega_1 s_1}{\omega_2 s_2} a_1(0) - \frac{V_0}{2\omega_2 s_2}, \tag{60}
\]
\[
c_1 a_1^2(0) + c_2 a_1(0) + c_3 = 0, \tag{61}
\]
where
\[
c_1 = \frac{q_{21} + q_{12}}{\Omega_2} + \frac{p_{22} \omega_1^2 s_1^2}{\Omega_1 \omega_2^2 s_2^2},
\]
\[
c_2 = -\frac{p_{22} \omega_1 s_1 V_0}{\Omega_1 \omega_2^2 s_2^2}, \quad c_3 = \frac{4\omega_2^2 V_0^2}{4\Omega_1 \omega_2^2 s_2^2} - \frac{V_0}{2}.
\]
Considering (56), from (45) we find the value of constant \( G_0 \)
\[
G_0 = \varepsilon \xi_0. \tag{62}
\]
Knowing constants \( E_0, G_0, \) and \( \xi_0 = \xi(0) \), we could determine the function \( \xi(T_1) \) from (47), while knowing \( \xi(T_1) \) we first define \( \cos \delta \) according to (44), and then substituting the known relationships for \( a_1(T_1), a_2(T_1) \) and \( \cos \delta \) in (33) and (35) after further integration over \( T_1 \) we could find \( \varphi_1(T_1) \) and \( \varphi_2(T_1) \). Thus, we have completely determined the values of \( w(t) \) and \( P(t) \).

Constant \( G_0 \) defines the trajectory of a point on the phase plane, while constant \( E_0 \) governs the velocity of motion of this point along the chosen trajectory, i.e., these constants completely determine the vibratory motions of the domain of contact of the shell with the impactor during the process of impact.

Since the process of impact is a snap-action process and the functions \( \xi(T_1), \varphi_1(T_1) \) and \( \varphi_2(T_1) \) depend on the slow time \( T_1 = \varepsilon t \), then it is possible to represent the enumerated functions from (47), (33) and (35) in the form:
\[
\xi(T_1) = \xi_0 - B_1 T_1, \tag{63}
\]
\[
\varphi_1(T_1) = -\frac{\pi}{2} + B_2 T_1, \tag{64}
\]
\[
\varphi_2(T_1) = \frac{\pi}{2} + B_3 T_1, \tag{65}
\]
where
\[
B_1 = b\sqrt{\xi_0^2 - (2 + e^2)\xi_0^2 + (1 + 2e G_0)\xi_0 - G_0^2},
\]
\[
B_2 = -\frac{1}{2} \sigma_1, \quad B_3 = -\frac{1}{2} \sigma_2.
\]

4.2 The contact force

Now knowing \( a_1(0), a_2(0), \varphi_1(0), \) and \( \varphi_2(0), \) it is possible to calculate the value \( P(t) \), which within an accuracy of \( \varepsilon \) has the form:
\[
P(t) = -M \varepsilon \left[ \dot{X}_1(t) s_1 + \dot{X}_2(t) s_2 \right] + O(\varepsilon^2), \tag{66}
\]
or with due account for (53)
\[
P(t) = 2M \varepsilon \left[ s_1 \Omega_1^2 a_1(0) \cos [\Omega_1 t + \varphi_1(0)]
\right.
\]
\[
+ s_2 \Omega_2^2 a_2(0) \cos [\Omega_2 t + \varphi_2(0)] \right) + O(\varepsilon^2). \tag{67}
\]

Considering (58) and (23), Eq. (67) is reduced to
\[
P(t) = 2M \varepsilon \left( s_1 \Omega_1^2 a_1(0) \sin \Omega_1 t - s_2 \Omega_2^2 a_2(0) \sin \Omega_2 t \right)
\]
\[
= 4M \varepsilon s_1 \Omega_1^2 a_1(0) \sin \Omega_2 t \left( \cos \Omega_2 t - \frac{1}{8} \alpha \right)
\]
\[
+ O(\varepsilon^2), \tag{68}
\]
where \( \alpha = \frac{q_{22}(0) s_2}{a_1(0) s_1}. \)

The contact force in the dimensionless form could be written as
\[
P^*(t) = \sin \Omega_2 t \left( \cos \Omega_2 t - \frac{1}{8} \alpha \right), \tag{69}
\]
where \( P^*(t) = P(t) \left( 4\varepsilon M s_1 \Omega_2^2 a_1(0) \right)^{-1} \).
by (69) is shown in Figure 2 for the different magnitudes of the parameter $\varepsilon$: 0.008, 1, 2, and 4, when $2\Omega_{21} \approx \Omega_{33}$ according to the data from Table 1 in [1].

Reference to Figure 2 shows that the decrease in the parameter $\varepsilon$ results in the increase of both the maximal contact force and the duration of contact.

4.3 Particular case

The combination of two interacting modes and the coordinates of the point of impact could result in some particular cases, when $s_1$ or $s_2$ is equal to zero. As an example, we consider the case, when $s_1 \neq 0$, while $s_2 = 0$. Then Eqs. (60), (61) and (54) are reduced to

\[
\begin{align*}
    a_1(0) &= \frac{V_0}{2\Omega_1 s_1}, & \cos \varphi_1(0) &= 0, & \varphi_1(0) &= \frac{\pi}{2}, \\
    a_2(0) &= 0, & \cos \varphi_2(0) &= 0.
\end{align*}
\]

(70)

(71)

Thus, with due account for (70) and (71), the contact force is defined as

\[ P(t) = \varepsilon MV_0 \Omega_1 \sin \Omega_1 t. \]  

(72)

5 Conclusion

In the present paper, a new approach has been proposed for the analysis of the impact interactions of nonlinear doubly curved shallow shells with rectangular base under the low-velocity impact by an elastic sphere. It has been assumed that the shell is simply supported and partial differential equations have been obtained in terms of shell’s transverse displacement and Airy’s stress function, in so doing the local bearing of the shell and impactor’s materials was neglected with respect to the shell deflection in the contact region. The equations of motion have been reduced to a set of infinite nonlinear ordinary differential equations of the second order in time and with cubic and quadratic nonlinearities in terms of the generalized displacements. The contact force was considered to be a small value. Assuming that only two natural modes of vibrations dominate during the process of impact and applying the method of multiple time scales, the set of equations has been obtained, which allows one to investigate different possible types of internal resonance and to find the time dependence of the contact force.

From the comparison of formulas (68) and (72) corresponding to the cases of two-to-one internal resonances, it is evident that the time dependence of the contact force depends essentially on the position of the point of impact. The intricate $P(t)$ dependence at the two-to-one internal resonance (68) gives way to the simple sine dependence (72), what is an accordance with a priori assumption of some researchers about a sine character of the contact force variation with time [18]–[20].

Besides, the contact force depends essentially on the magnitude of the initial energy of the impactor. This value governs the place on the phase plane, where a mechanical system locates at the moment of impact, and the phase trajectory, along which it moves during the process of impact.

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References:


