An impulse-based algorithm for dynamical systems with multiple simultaneous collisions.

RAZVAN ANDREI OPREA¹, CORNELIA STAN²
¹Railway Vehicles Engineering Department
²Automotive Engineering Department
University Politehnica of Bucharest
Splaiul Independenţei nr. 313, sector 6, Bucureşti
ROMANIA
razvan.oprea@upb.ro

Abstract: - Simultaneous multiple collisions of independent bodies is a most improbable event. However, an equivalent occurrence, the collision of an individual object with a group of objects in contact is quite generic. The ill-posed nature of simultaneous collisions is not a mathematical fault or a consequence of an unsuitable modeling approach. Instead, it proves to be a physical property of the phenomena. In this paper, an impulse-based algorithm for simultaneous collision resolution is embedded into an equality of measures. The coefficient of restitution is set to 1 for each pair of bodies after the first impact in the sequences which substitute simultaneous collision. This approach is a novelty in the general scheme of the impulse-based algorithm. Both impulsive and non-impulsive dynamics are casted in a general framework for the study of the systems which feature simultaneous multiple collisions. The numerical solution is achieved using a time-stepping algorithm. An application example points out the capability of the method.

Key-Words: - multiple impacts, time-stepping algorithm, non-smooth dynamics, measure differential inclusion, coefficient of restitution

1 Introduction
Simultaneous multiple collisions of independent bodies is a most improbable event. However, the collision of an individual object with a group of objects in contact is quite generic. This situation often arises in mechanisms and a choice has to be made whether it will be treated as a set of bodies or as an intact mechanism. The latter option leads to a single direct calculation collision but it is prone to errors since the mechanism constraints may not be observed during the collisions. The alternative is to approach the phenomenon as simultaneous multiple collisions.

A paper published no later than 1952 [1] is the first to argue that the usual laws of impact are unable to determine a unique outcome in the case of simultaneous collisions. A series of examples analyze a set of balls constrained to move in a straight line. The balls experience simultaneous collisions in both versions previously recalled. The final velocities prove to be discontinuous functions either of the initial positions of the balls or of the masses.

The conclusion was that determinism, abandoned in the field of atomic physics, had to fall down also in the classical case. The ill-posed nature of simultaneous collisions is not a mathematical fault or a consequence of an unsuitable modeling approach, and it proves to be a physical property of the collisions. There are classical examples of this indeterminacy, probably the most vivid one being the outcome of a “pool break” which varies from a game to another and may not be reproduced or controlled.

In order to get a complete description of the impact dynamics, additional impact laws have to be introduced. The utility of such laws are not only to predict the resulting velocities of a collision but also to be able to use experimental data concerning the bodies studied. For the multi-contact case, in particular, inequalities have to be formulated to describe the impact process [2] and to constrain the resulting impulses to remain in a permissible region [3].

The restrictions regarding the outcome of a collision issue from natural assumptions like non-interpenetration in the final collision velocity, which
is a fundamental condition, and others related to the kinetic energy dissipation or the resulting impulses.

Obviously, any pair of objects which collide must satisfy the classical Newton kinematic impact law, which is used in this work to predict the outcome of a simple collision. However, this does not limit the validity of the method proposed to this particular formulation as alternate formulation may be used instead, see for instance [2, 3].

In this paper algorithm for simultaneous collision resolution is embedded into an equality of measures to cast both impulsive and non-impulsive dynamics in a general framework for the study of the systems which feature simultaneous multiple collisions. The conjunction of these formulations yields a measure differential inclusion (MDI) which describes the system time evolution at the velocity level [4]. In addition to the general benefits of the MDI approach, a particular advantage for the present application is that contact closing is a natural result of the integration.

The numerical solution is achieved using a time-stepping algorithm. Each integration step implies a contact-impact problem which includes a preliminary step regarding impulsive forces resolution. Contact closings (or captures) are preceded by impact accumulation points. An application example points out the capability of the method.

2 Newton restitution law and algorithms for simultaneous collisions

As the aim of the present work is to develop a general framework for the study of multiple impacts, the classical Newton kinematic impact law will be used to predict the outcome of a simple collision, which involves only two bodies which have the masses \( M_i \) and \( M_{i+1} \), Equation 1, where \( u^- \) and \( u^+ \) are the velocities before and after the collision, respectively.

\[
M_i (u_i^+ - u_i^-) + M_{i+1} (u_{i+1}^+ - u_{i+1}^-) = 0 \tag{1a}
\]

\[
\dot{g}_i^- - \epsilon \dot{g}_i^+ = 0 \tag{1b}
\]

The above system of equation comprises the law of the conservation of momentum (1a), and the definition of the coefficient of restitution (COR), whose usual value is \( 0 \leq \epsilon \leq 1 \), (1b). The zero magnitude corresponds to a perfectly inelastic collision, at which the relative velocity \( \dot{g}^- \) vanishes, whereas the magnitude 1 represents a perfectly elastic one.

Moreover, the impact equation which describes the impulsive motion may be formulated introducing the impulsive forces \( P \) as in Equation (2)

\[
M(u^+ - u^-) - W_p P = 0 \tag{2}
\]

where \( M \in \mathbb{R}^{n \times n} \) is the inertia matrix, \( M = diag(M_i), i \in \{1, ..., n\} \), and \( W_p \in \mathbb{R}^{n \times m} \) the corresponding constraint matrix which describes the action of the impulsive forces.

Impulsive forces, \( P \), can be described by set-valued laws [5]. When two bodies close in and the gap \( g \) is covered, a rebound may take place and the relative velocity would become positive. The forces involved in common collisions act as compressive magnitudes which preclude the closing bodies from interpenetrating. Therefore, impulsive force may be defined as in Equation (3).

\[
P \in \left\{ \begin{array}{ll}
\{0, \infty\}, & \text{if } g = 0 \land \dot{g} < 0, \\
0, & \text{else}
\end{array} \right. \tag{3}
\]

An impact occurs between two bodies which come into contact and the relative velocity does not vanish. Although a simultaneous impact of a larger group of bodies is improbable in the above defined sense, it is most plausible that a group of bodies in contact may collide to another body (or group). Such an event may be regarded as a simultaneous multiple collision, an indeterminate problem in classical mechanics [1].

An intrinsic attribute of simultaneous collisions is the ill-posed nature of the problem [6]. The ill-posedness is a fundamental property of this phenomenon and cannot be removed by any formalism. Therefore, the main goal of impacts studies is to achieve an efficient and consistent analysis framework [7].

The set-valuedness of the impulsive forces may yield indefinite values. However, decomposing the simultaneous collision into a sequence of impacts provides a solution, as described in the following section. The successive impacts are solved using the general Newton restitution law, Equation (1).

3. The dynamics with impacts problem

The classical Newton-Euler equations which describe unconstrained motion of a multibody system with \( n \) degrees of freedom may be stated as:

\[
M \ddot{u} - h(t, q, u) = 0 \tag{4a}
\]

\[
\dot{q} = u \tag{4b}
\]
where \( q \in \mathbb{R}^n \) is the generalized coordinates vector, \( u \in \mathbb{R}^n \) the generalized velocities vector and \( h \in \mathbb{R}^n \) is the sum of the forces described by constitutive laws – the inertia matrix \( M \).

Further, the impact equation which describes the impulsive motion may be formulated introducing the impulsive forces \( P \) as in Equation (2) Using an equality of measures [4], the impulsive and non-impulsive dynamics may be embraced in a general formulation. The motion and impact equations (4) and (2) may be multiplied with the Lebesgue measure \( dt \) and the atomic measure \( d\eta \) (i.e., the sum of the Dirac point measures at the collisions) and they may be added to yield Equation (5). In this framework, the numerical solution may be accomplished within a unitary integration process [12].

\[
M[\dot{u}dt + (u^+ - u^-)d\eta] - hdt - W_Pd\eta = 0
\]

(5)

Constitutive forces laws are included in the vector \( h \). The dynamics of a system with impacts is completely described by the equality of measures (5) and by the set-valued law of the impact (3). The equation system forms a measure differential inclusion which may be solved using time-stepping methods. Each integration step requires the resolution of the impulsive forces.

3.1. The time-stepping algorithm

A distinctive feature of the bodies collision study is the transition from the relative motion to a closed contact (capture) through a sequence of shocks, i.e., an impact accumulation point. This transition is automatically treated by time-stepping algorithms and, therefore, this type of numerical scheme will be employed in the study.

Time-stepping methods are most simple and robust [2, 13, 14]. Moreover, they are a real difference equations approach and they are based on velocity constraints, thus reducing the system index. In the present work, the Non-smooth Contact Dynamics (NSCD) method [4] will be considered. The numerical scheme is very general and is proved to converge globally for \( 0 \leq e \leq 1 \) , [15]. The drawback of the method is that it uses a first order discretization, but in the case of the studied case, this is not relevant as the system undergoes many collisions and after each impact, the order decreases to one [16].

The integration step begins with the computation of the midpoint time instant \( t_m \), the corresponding position state vector \( q_m \) and the constitutive forces vector \( h_m \).

\[
t_m = t_a + 1/2\Delta t
q_m = q_a + 1/2u_a\Delta t
h_m = h(t_m, q_m, u_a)
\]

(6)

2. Collision detection and impulse resolution, which is accomplished analyzing the vectors \( g \) and \( g \) as indicated in (3).

3. Update the state vector.

\[
u_b = u_\alpha + M^{-1}(h_m\Delta t + W_P)
q_b = q_m + 1/2u_b\Delta t
\]

(7)

3.2 Impulsive forces resolution

The constraint matrices relate generalized coordinates and forces to the contact space. The gap and the relative velocity in the contact points are consequently given by (8).

\[
g = W_P^Tq_m
\]

(8a)

\[
\dot{g} = W_P^T\dot{q}_m
\]

(8b)

The detection of the impacts and phase transitions instants is one of the problems related to the simulation of systems with collisions. A review of the accurate methods to locate zero-crossings may be found in [17], but, as a general feature, they are very expensive in terms of the required computational power. Besides, machine epsilon impedes maintaining numerically the zero level. A simple solution is given by the Karnopp model [16] which formulates the zero velocity condition as \(|q| < \eta \) (instead of \( q = 0 \)).

As explained above, the collision resolution precedes the constitutive forces computation. In the next stage of each integration step, possible impacts are sought verifying the conditions expressed in (3). If common or simultaneous collisions occur, the
impact of the pair of bodies with the highest magnitude relative velocity is resolved using (1) and ignoring the other collisions. The first collision for each pair of bodies which occurs in a sequence is solved using the specified COR. If, however, subsequent impacts of the same couple of bodies occur, they will be solved considering that the COR is 1. The set of the resulting relative velocities is examined and the process is repeated until equation (3) indicates motion without impulses for all contacts. Finally, the state vector may be updated. The integration step stages are illustrated in Figure 1. Practical solutions to the numerical instabilities which can occur when solving such systems are available, e.g. [18], and they can be embeded in the scheme.

4 Application example
An application example is described in the following. The numerical experiment takes into account six rigid bodies constrained to move only along the vertical axis. The bodies can encounter central collisions and they can stack one on top of the other. The example analyzed assumes that five bodies are at rest and the 6th collides on top as in Figure 2. The system parameters are described in Table 1. Impulsive forces are applied between each pair of bodies which may come into contact and also between the body at the bottom and the ground. The corresponding constraint matrix is given in Equation 9.

The outcome of the impact and the time history of the bodies positions are plotted in Figure 3. Several simultaneous impacts take place, most of
them similar with the initial collision, when a falling body hits a few others at rest. After a few instants all the bodies rest in closed contact. A couple of similar closing contacts phases occur during the entire simulation. Such events are preceded by accumulation points.

\[
W_p = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & \ldots & 0 \\
0 & -1 & 1 & \ddots & \vdots \\
& \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & -1 & 1 & 0 \\
0 & -1 & 1 & 
\end{bmatrix}
\]  

(9)

In order to compare the results obtained using the method proposed in this work, the outcome of the simultaneous collisions was alternatively computed using a “classical” impulse algorithm, where the COR value is kept constant along the entire sequence of impacts, Figure 4. It is obvious that in this latter case the bodies have considerably lower velocities after simultaneous collisions.

Fig. 2 A system of bodies constrained to move along the vertical axis.
Fig. 3 Time history of the bodies positions. COR is modified as proposed in Section 2.1.

Fig. 4 Time history of the bodies positions. COR is kept constant – “classical” approach.

Table 1 Parameter values for the colliding bodies model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>14</td>
<td>Mass of body 1 (top)</td>
<td>kg</td>
</tr>
<tr>
<td>$m_2$</td>
<td>2</td>
<td>Mass of body 2</td>
<td>kg</td>
</tr>
<tr>
<td>$m_3$</td>
<td>6</td>
<td>Mass of body 3</td>
<td>kg</td>
</tr>
<tr>
<td>$m_4$</td>
<td>3</td>
<td>Mass of body 4</td>
<td>kg</td>
</tr>
<tr>
<td>$m_5$</td>
<td>2</td>
<td>Mass of body 5</td>
<td>kg</td>
</tr>
<tr>
<td>$m_6$</td>
<td>4</td>
<td>Mass of body 6 (bottom)</td>
<td>kg</td>
</tr>
<tr>
<td>$e$</td>
<td>0.9</td>
<td>COR for all potentially colliding bodies</td>
<td></td>
</tr>
<tr>
<td>$v_0$</td>
<td>-5</td>
<td>Initial velocity of the top body</td>
<td>m/s</td>
</tr>
</tbody>
</table>

5 Conclusions

The present work proposes an impulse-based algorithm which has the ability to integrate accumulation points and keeps control on the system energy. A unitary, consistent and reliable framework is accomplished by a MDI. The equation is formulated at the velocity level and it casts both impulsive and non-impulsive dynamics. The solution is obtained with the NSCD time-stepping algorithm. The model formulation has strong mathematical foundations. The time stepping algorithm is a real MDI approach, best suited for the convergence analysis.

Simultaneous impacts are dealt with an iterative algorithm. An appropriate numerical scheme is designed to integrate the system. As a result, the simulation is remarkably simple and efficient.
A novelty in the general scheme of the impulse based algorithm is the adjustment of the COR to preserve the energy balance of the system. The existence of the solution for the simultaneous multiple collisions algorithm is guaranteed. Therefore, the approach described in the present work definitely provides a solution.

A demonstrative application illustrates the capabilities and benefits of the scheme. Distinctive non-smooth dynamics phenomena are revealed. The algorithm does not imply any special restrictions regarding the system features or the number of degrees of freedom. Therefore, it may be used for a large class of physical and engineering problems.

References: