Approximate formulas for the Point-to-Ellipse and for the Point-to-Ellipsoid Distance Problem

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Abstract: We deduce some approximate formulas for the distance evaluation problem between a point and an ellipse in $\mathbb{R}^2$ and a point and an ellipsoid in $\mathbb{R}^3$. The proposed approach is based on the analytical representation of the distance function as a zero of an appropriate univariate algebraic equation.

Key–Words: Distance, point-to-ellipse, point-to-ellipsoid

1 Introduction

The problem of finding the value of the distance from the given point X_0 to the given quadric in $\mathbb{R}^n$ is of great importance for several branches of mathematics, statistical data analysis, astronomy, particle physics and image processing.

The existed approaches for solving this problem can be sorted to be either numerical or analytical. The first one consists in construction of an appropriate iterative procedure for solving the nonlinear constrained optimization problem: the aim is to generate the sequence converging to the nearest to X_0 point in the quadric. This might be effective for the number of problems when the treated quadric is assumed to be *fixed*, i.e. when all of its coefficients are specialized. However, in several industrial applications, the parameters of the quadric may *variate*, such as, for instance, when it simulates an object moving in $\mathbb{R}^3$. To solve the problem in such a statement, it is needed an analytical expression either for the distance or for its suitable approximation as a function of parameters.

The necessity in the analytical (symbolical) representation is also stemmed from the problem of approximation of the scattered data known as the *ellipse* (*ellipsoid*) *fitting problem* [1], [2], [3], [4]. The latter consists in finding the coefficients of the second order algebraic equation

$$G(X) \stackrel{def}{=} X^T\mathbf{A}X + 2\,B^TX + c = 0 \qquad (1)$$

providing the ellipse (or ellipsoid) closest to the given set of test (measured) data points $\{X_j\}_{j=1}^N \subset \mathbb{R}^n$. The first obstacle in solving this problem consists in evaluation of the closeness of the given point X_j to the quadric (1) with *undetermined coefficients*. Since the explicit formula for the distance function is unavailable, this closeness is usually evaluated by simpler computed substitutes, like, for instance, the *algebraic distance* $|G(X_j)|$ or $G^2(X_j)$. The effectiveness of such substitution has been evaluated empirically; the error analysis has never been carried out.

The authors of the present paper have succeeded in finding the expression for the "true", i.e. Euclidean, distance from the point to the quadric (1) in $\mathbb{R}^n$ [5], [6]. The result has been achieved via the application of analytical (algebraic) methods of *elimination of variables* from the system of equations providing the coordinates of stationary points of the constrained optimization objective function. Unfortunately, the obtained analytical expression for the distance function turns out to be *implicit*. The value of the distance is among the positive zeros of an appropriate univariate algebraic *distance equation* with the coefficients depending polynomially on the coefficients of the quadric (1) and the coordinates of X_0.

The aim of the present paper is to extract from this distance equation the explicit formulas for the distance function approximation and to estimate the tolerances for these approximations. Although some of the results considered below are valid for the general case of $\mathbb{R}^n$, we will be focused mainly on the problems in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Notation is kept to correlate with that from [5] and [6]. We set $c = -1$ in (1); X_0, B and X are treated as n-column vectors from $\mathbb{R}^n$, $\mathbf{A}$ is a symmetric sign-definite matrix of the order n, while $\mathbf{I}$ is the identity matrix of the same order. $\mathcal{D}_x$ denotes the discriminant of the polynomial (subscript denotes the variable w.r.t.

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which the polynomial is treated).

Remark 1 *In the rest of the paper we utilize essentially some basic results from Elimination Theory (dealing with the algebraic algorithms of elimination the variables from a system of nonlinear algebraic equations), including the notions of the* resultant *and the* discriminant*. To get acquainted with this theory, we refer the reader to the book [7] or to Section 2 of [6].*

2 Analytical Solution

In [6] the general result has been presented for finding the distance from a point to an ellipsoid in $\mathbb{R}^n$.

Theorem 2 *Let the point $X_0 \in \mathbb{R}^n$ not lie in the ellipsoid (1): $G(X_0) \neq 0$. The square of the distance from X_0 to the ellipsoid coincides with the minimal positive zero of the distance equation*

$$\mathcal{F}(\delta) \stackrel{def}{=} \mathcal{D}_\mu(\Phi(\mu,\delta)) = 0 \qquad (2)$$

provided that this zero is not a multiple one. Here

$$\Phi(\mu,\delta) \stackrel{def}{=} \det\left(\left[\begin{array}{cc} \mathbf{A} & B \\ B^T & -1 \end{array} \right] + \mu \left[\begin{array}{cc} -\mathbf{I} & X_0 \\ X_0^T & \delta - X_0^T X_0 \end{array} \right] \right);$$

Once the minimal positive zero δ_* of (2) is evaluated, one can find the value for the multiple zero μ_* for the polynomial $\Phi(\mu,\delta_*)$; it can be represented as a rational function of δ_* (with coefficients polynomially dependent on the coefficients of (1) and on coordinates of X_0). Then the coordinates of the nearest to X_0 point in the quadric (1) are as follows:

$$X_* = -\mathbf{A}^{-1}B - \mu_*(\mathbf{A}-\mu_*\mathbf{I})^{-1}(\mathbf{A}^{-1}B + X_0)\,.$$

We now intend to detail the above results for the particular cases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Corollary 3 *For the point $X_0 = (x_0, y_0)$ and the ellipse*

$$G(x,y) \stackrel{def}{=} x^2/a^2 + y^2/b^2 - 1 = 0 \qquad (3)$$

the distance equation can be constructed in the form (2) where

$$\Phi(\mu,\delta) = \mu^3 + A_1\mu^2 + A_2\mu + A_3\,.$$

Here

$$\begin{aligned} A_1 &= -a^2b^2\left\{ \frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} \right. \\ &\qquad \left. - \left(\frac{1}{a^2}+\frac{1}{b^2}\right) G(x_0,y_0) + \frac{\delta}{a^2b^2} \right\}, \\ A_2 &= a^2b^2\left\{ \left(\frac{1}{a^2}+\frac{1}{b^2}\right)\delta - G(x_0,y_0) \right\}, \\ A_3 &= -a^2b^2\delta\,. \end{aligned}$$

In terms of the coefficients of $\Phi(\mu,\delta)$, the expression for $\mathcal{F}(\delta)$ is as follows

$$\mathcal{F}(\delta) \equiv A_1^2A_2^2 - 4\,A_1^3A_3 - 4\,A_2^3 + 18\,A_1A_2A_3 - 27\,A_3^2$$

while its further expansion in powers of δ becomes rather cumbersome and we restrict ourselves here by presenting its leading and constant term

$$\begin{aligned} \mathcal{F}(\delta) &\equiv (a^2-b^2)^2\delta^4 + \ldots + a^{12}b^{12}G^2(x_0,y_0) \\ &\times \left\{ \left[\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} - \left(\frac{1}{a^2}+\frac{1}{b^2}\right)G(x_0,y_0) \right]^2 \right. \\ &\qquad \left. + \frac{4}{a^2b^2}G(x_0,y_0) \right\}. \end{aligned}$$

Once the minimal positive zero δ_* of this polynomial is evaluated, one can express the coordinates of the nearest to (x_0, y_0) point in the ellipse by:

$$x_* = \frac{a^2x_0}{a^2-\mu_*},\ y_* = \frac{b^2y_0}{b^2-\mu_*}\,. \qquad (4)$$

Here

$$\mu_* = \frac{9\,A_3 - A_1A_2}{2(A_1^2 - 3\,A_2)}$$

and substitution $\delta = \delta_*$ has been made into the expressions for A_1, A_2, A_3.

Corollary 4 *For the point $X_0 = (x_0, y_0, z_0)$ and the ellipsoid*

$$G(x,y,z) \stackrel{def}{=} x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0 \quad (5)$$

the distance equation can be constructed in the form (2) where

$$\Phi(\mu,\delta) = \mu^4 + A_1\mu^3 + A_2\mu^2 + A_3\mu + A_4\,.$$

Here

$$\begin{aligned} A_1 &= x_0^2 + y_0^2 + z_0^2 - \delta - a^2 - b^2 - c^2, \\ A_2 &= a^2b^2c^2\left\{ \left(\frac{1}{b^2c^2} + \frac{1}{a^2c^2} + \frac{1}{a^2b^2}\right)\delta \right. \end{aligned}$$

$$
\begin{aligned}
& + \left(\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4} \right) \\
& - \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) G(x_0, y_0, z_0) \Big\}, \\
A_3 &= a^2 b^2 c^2 \Big\{ G(x_0, y_0, z_0) \\
& - \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \delta \Big\} \\
A_4 &= a^2 b^2 c^2 \delta \, .
\end{aligned}
$$

In terms of the coefficients of $\Phi(\mu, \delta)$, the expression for $\mathcal{F}(\delta)$ is as follows

$$\mathcal{F}(\delta) \equiv 4 I_2^3 - 27 I_3^2$$

where

$$
\begin{aligned}
I_2 &\stackrel{def}{=} 4 A_4 - A_1 A_3 + \frac{1}{3} A_2^2, \\
I_3 &\stackrel{def}{=} -A_3^2 - A_1^2 A_4 + \frac{8}{3} A_2 A_4 + \frac{1}{3} A_1 A_2 A_3 \\
& \quad - \frac{2}{27} A_2^3 \, .
\end{aligned}
$$

Further expansion of the distance polynomial in powers of δ is again represented by its leading and constant term:

$$
\begin{aligned}
& \mathcal{F}(\delta) \equiv (a^2 - b^2)^2 (a^2 - c^2)^2 (b^2 - c^2)^2 \delta^6 + \dots \\
& + a^4 b^4 c^4 G^2(x_0, y_0, z_0) \mathcal{D}_\mu (\mu^3 + A_1 \mu^2 + A_2 \mu + A_3) \, ,
\end{aligned}
$$

here substitution $\delta = 0$ has been made into the expressions for A_1, A_2, A_3. Once the minimal positive zero δ_* of this polynomial is evaluated, the coordinates of the nearest to (x_0, y_0, z_0) point in the ellipse can be expressed as:

$$x_* = \frac{a^2 x_0}{a^2 - \mu_*}, \; y_* = \frac{b^2 y_0}{b^2 - \mu_*}, z_* = \frac{c^2 z_0}{c^2 - \mu_*} \, . \quad (6)$$

Here

$$\mu_* = \frac{2 A_1 I_2^2 + (3 A_1 A_2 - 18 A_3) I_3}{(24 A_2 - 9 A_1^2) I_3 - 8 I_2^2} \quad (7)$$

and substitution $\delta = \delta_*$ has been made into the expressions for A_1, A_2, A_3, A_4.

Remark 5 *The results of the present section can be extended to the problem of finding* the farthest *distance from X_0 to the point in the ellipse or ellipsoid. For this aim, in any of the above presented results connected with the distance equation and expressions for the nearest point coordinates, one should take δ_* to be* the greatest *positive zero for the corresponding distance equation.*

Example 6 *Find the nearest and the farthest point in the ellipsoid $x^2/4 + y^2/16 + z^2/49 = 1$ to the point* $(6, -2, 5)$.

Solution: Compute the distance equation via Corollary 4:

$$
\begin{aligned}
& 19847025 \, \delta^6 - 8393808060 \, \delta^5 + 1317736785456 \, \delta^4 \\
& -103262746605120 \, \delta^3 + 4327358033988864 \, \delta^2 \\
& -91883501048862720 \, \delta + 757148717189025792 = 0 \, .
\end{aligned}
$$

It has exactly two positive zeros, namely

$$\delta_1 \approx 21.63634, \delta_2 \approx 186.72961 \, .$$

Therefore the distance from the point to the ellipsoid equals $\sqrt{\delta_1} \approx 4.65149$. Compute the multiple zero for polynomial $\Phi(\delta_j, \mu)$ by formula (7):

$$\mu_1 \approx -11.70096, \; \mu_2 \approx 84.64247 \, .$$

Formulas (6) give one the coordinates of the nearest

$$X_1 \approx (1.52857, -1.15519, 4.03618),$$

and the farthest

$$X_2 \approx (-0.29761, 0.46618, -6.87382)$$

point in the considered ellipsoid.

3 Approximate Solution

In the previous section we have deduced an analytical solution for the distance evaluation problem. The stated problem is reduced to that of solving the distance equations. For any specialization of the parameters (i.e. the given point coordinates and the coefficients of the ellipse or ellipsoid equations), this can be done numerically. However, to resolve these equations analytically and to get an explicit expression for the distance as a function of the parameters is not a trivial task. One of the possible approaches for finding the value of a zero of an algebraic equation in terms of its coefficients consists in finding an expansion of the zero in an appropriate power series. Let us choose the latter to be the one in powers of the value $G(X_0)$, since this value, in a vicinity of the considered ellipse or ellipsoid, can be treated as a small parameter. We restrict our treatment with the first three terms of this expansion:

$$\delta = A_1 G(X_0) + A_2 G^2(X_0) + A_3 G^3(X_0) + \dots$$

To determine the coefficients A_1, A_2, A_3, substitute this expansion into corresponding distance equation,

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expand the result in powers of $G(X_0)$ and equate the coefficients of $G(X_0), G^2(X_0), G^3(X_0)$ to zero. For the case of the ellipsoid (5), the result is as follows: $A_1 = 0$,

$$A_2 = \frac{1}{4}\frac{1}{(x_0^2/a^4 + y_0^2/b^4 + z_0^2/c^4)}, \quad (8)$$

$$A_3 = \frac{1}{8}\frac{x_0^2/a^6 + y_0^2/b^6 + z_0^2/c^6}{(x_0^2/a^4 + y_0^2/b^4 + z_0^2/c^4)^3}; \quad (9)$$

while the counterparts of these formulas for the case of planar ellipse (3) can be obtained by setting $z_0 = 0$.

Once the two approximations for the distance are obtained for quadric given by canonical equation, one can extend these results to the case of a quadric represented in the form (1).

Theorem 7 *The following formulas can be utilized for the distance approximation from the point* $X_0 \ne -\mathbf{A}^{-1}B$ *to the quadric (1) in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$*:*

$$d_{(1)} \stackrel{def}{=} \frac{|G(X_0)|}{2\sqrt{(\mathbf{A}X_0 + B)^T(\mathbf{A}X_0 + B)}} \quad (10)$$

and

$$d_{(2)} \stackrel{def}{=} d_{(1)} \quad (11)$$
$$\times\sqrt{1 + \frac{(\mathbf{A}X_0 + B)^T\mathbf{A}(\mathbf{A}X_0 + B)}{2\left((\mathbf{A}X_0 + B)^T(\mathbf{A}X_0 + B)\right)^2}G(X_0)}.$$

Proof: We restrict ourselves here with the proof of formula (10) for the case $\mathbb{R}^3$. Let us start with the case when $B = \mathbf{0}$; thus matrix $\mathbf{A}$ is assumed to be positive definite. The distance from the point X_0 to the ellipsoid $X^T\mathbf{A}X = 1$ is unaltered under the transformation $Y = QX$ with an orthogonal matrix Q: it equals to the distance from $Y_0 = QX_0$ to $Y^TQ\mathbf{A}Q^TY = 1$. Choose this transformation with the aim to reduce the ellipsoid to canonical form:

$$Q\mathbf{A}Q^T = \mathbf{A}_{diag} = \begin{pmatrix} 1/a^2 & 0 & 0 \\ 0 & 1/b^2 & 0 \\ 0 & 0 & 1/c^2 \end{pmatrix}.$$

Here $1/a^2, 1/b^2, 1/c^2$ stand for the eigenvalues of the matrix $\mathbf{A}$. Apply the representation (8):

$$d_{(1)}^2 = A_2(Y_0^T\mathbf{A}_{diag}Y_0 - 1)^2 = \frac{(Y_0^T\mathbf{A}_{diag}Y_0 - 1)^2}{4\,Y_0^T\mathbf{A}_{diag}^2Y_0}$$
$$= \frac{(Y_0^TQ\mathbf{A}Q^TY_0 - 1)^2}{4\,Y_0^T(Q\mathbf{A}Q^T)^2Y_0} = \frac{(Y_0^TQ\mathbf{A}Q^TY_0 - 1)^2}{4\,Y_0^TQ\mathbf{A}^2Q^TY_0}$$
$$= \frac{(X_0^T\mathbf{A}X_0 - 1)^2}{4\,X_0^T\mathbf{A}^2X_0}.$$

The case when $B \ne \mathbf{0}$ can be reduced to the previous one with the aid of transformation

$$X = Y + X_c, \text{ where } X_c \stackrel{def}{=} -\mathbf{A}^{-1}B$$

denotes the ellipsoid center. The equation of the ellipsoid takes now the form

$$Y^T\tilde{\mathbf{A}}Y = 1 \text{ with } \tilde{\mathbf{A}} \stackrel{def}{=} -\frac{\mathbf{A}}{G(X_c)} = \frac{\mathbf{A}}{B^T\mathbf{A}^{-1}B + 1}.$$

□

It is possible to provide the following geometric interpretation for the approximation (10). This value coincides with the distance from the point X_0 to the linear manifold obtained by linearization of (1) at the point X_0:

$$G(X_0) + \left.\frac{D\,G}{D\,X}\right|_{X=X_0}(X - X_0) = 0.$$

Here the n-row $D\,G/D\,X|_{X=X_0}$ stands for the gradient of the function $G(X)$ calculated in the point X_0. This formula was suggested in [1] for the distance approximation from a point to arbitrary algebraic curve or surfaces in $\mathbb{R}^n$. However no estimation has been suggested since then for the accuracy of this approximation. For the case of quadric, one might definitely expect the problems with this when the point X_0 is close to the quadric center.

4 Error Estimation

To estimate the error of the distance approximation by formulas (10) or (11) we suggest to fix the values of these approximations and estimate the maximal and minimal deviations for the points in the obtained manifolds from the quadric (1).

We restrict ourselves here by the planar case of the ellipse represented in canonical form (3) and with the distance approximation given by (10). Consider the level curves of the function $d_{(1)}^2(x, y)$; they can be represented as algebraic curves:

$$K_h(x, y) \quad (12)$$
$$\stackrel{def}{=} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2 - 4\,h\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) = 0;$$

here the parameter $h > 0$ has the meaning of the squared "approximate" distance. We will call by the *maximal* (or the *minimal*) *deviation* of the curve (12) from the ellipse (3) the maximal (or, respectively, the minimal) distance between all the pairs of the nearest points in these curves.

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Theorem 8 *Let $a \neq b$. The values of the minimal and the maximal deviation of (12) from the ellipse (3) are either among the zeros of the equations*

$$\delta^2+2\,a\delta\pm\sqrt{h}(\delta+a)=0\,,\ \delta^2+2\,b\delta\pm\sqrt{h}(\delta+b)=0 \quad (13)$$

or among the square roots of the positive zeros of the equation

$$\mathcal{F}(\delta,h)\stackrel{def}{=}C_0\delta^4+C_1\delta^3+\ldots+C_4=0\,. \quad (14)$$

Here

$$\begin{array}{rcl} C_0 &=& a^4b^4(a^2-b^2)^2, \\ C_1 &=& -2\,a^2b^2(a^8+b^8-4\,a^4b^4)h \\ && -2\,a^4b^4(a^2+b^2)(2\,a^4-5\,a^2b^2+2\,b^4), \\ C_2 &=& (a^8-10\,a^4b^4+b^8)(a^2+b^2)^2h^2 \\ && +6\,a^4b^4(a^2+b^2)(a^4-7\,a^2b^2+b^4)h \\ && -27\,a^8b^8, \\ C_3 &=& 2\,a^2b^2h\left\{2\,(a^2+b^2)h+3\,a^2b^2\right\} \\ && \times\left\{(a^2+b^2)^3h+9\,a^4b^4\right\}, \\ C_4 &=& -a^4b^4h^2\left\{4\,(a^2+b^2)^3h+27\,a^4b^4\right\}\,. \end{array}$$

Proof: We set the problem of constrained optimization: find the critical values (including the maximal and the minimal ones) of the function

$$(x-\tilde{x})^2+(y-\tilde{y})^2$$

subject to

$$K_h(x,y)=0,\ G(\tilde{x},\tilde{y})=0\,.$$

For its solution, we utilize the Lagrange multipliers method. We first construct the Lagrange function

$$(x-\tilde{x})^2+(y-\tilde{y})^2-\mu_1K_h(x,y)-\mu_2G(\tilde{x},\tilde{y})$$

and then equate to zero its derivatives with respect to the variables $x,y,\tilde{x},\tilde{y},\mu_1,\mu_2$:

$$\begin{array}{rcll} x-\tilde{x} &=& \dfrac{2\,\mu_1x}{a^2}\Big(G(x,y)-2h/a^2\Big), & (15) \\ y-\tilde{y} &=& \dfrac{2\,\mu_1y}{b^2}\Big(G(x,y)-2h/b^2\Big), & (16) \\ x-\tilde{x} &=& -\mu_2\tilde{x}/a^2, & (17) \\ y-\tilde{y} &=& -\mu_2\tilde{y}/b^2, & (18) \\ K_h(x,y) &=& 0, & (19) \\ G(\tilde{x},\tilde{y}) &=& 0\,. & (20) \end{array}$$

Complement the obtained system with the equation

$$\delta=(x-\tilde{x})^2+(y-\tilde{y})^2 \quad (21)$$

which introduces a new variable responsible for the critical values of the distance function. Our aim is to eliminate all the variables from the system (15) – (21) except for δ. To do this, we express $\tilde{x}$ and $\tilde{y}$ from (17) and (18); the result is similar to (4):

$$\tilde{x}=\frac{a^2x}{a^2-\mu_2},\quad \tilde{y}=\frac{b^2y}{b^2-\mu_2}\,. \quad (22)$$

Substitute these values into (20) and (21):

$$\begin{array}{rcll} \dfrac{a^2x^2}{(a^2-\mu_2)^2}+\dfrac{b^2y^2}{(b^2-\mu_2)^2}-1 &=& 0, & (23) \\ \dfrac{\mu_2^2x^2}{(a^2-\mu_2)^2}+\dfrac{\mu_2^2y^2}{(b^2-\mu_2)^2}-\delta &=& 0 & (24) \end{array}$$

Substitute next (22) into (15) and (16); the resulting equations can be splitted into the alternatives:

$$x=0 \quad \text{or} \quad \frac{\mu_2}{\mu_2-a^2}=\frac{2\mu_1}{a^2}\left(G(x,y)-\frac{2\,h}{a^2}\right);$$

$$y=0 \quad \text{or} \quad \frac{\mu_2}{\mu_2-b^2}=\frac{2\mu_1}{b^2}\left(G(x,y)-\frac{2\,h}{b^2}\right).$$

The first parts of these alternatives correspond to the equations (13). From the second parts, it is possible to eliminate the parameter μ_1 and to find the expression for μ_2:

$$\mu_2=\frac{2\,h}{2\,h/a^2+2\,h/b^2-G(x,y)}\,. \quad (25)$$

Substitute this expression into (23) and (24). The obtained equations depend now on the variables x,y and δ. Together with equation (19), these equation constitutes the system in x,y and δ. Convert its equations into algebraic form and eliminate the variables x,y via the resultant computation. On excluding the extraneous factor, the resulting equation coincides with (14). □

Example 9 *Find the maximal and the minimal deviations of the curve $K_1(x,y)=0$ from the ellipse*

$$x^2/196+y^2/4=1\,.$$

Solution: The equation (14) is (up to the factor 2^{14}), as follows:

$$1382976\,\delta^4-1235618251\,\delta^3+4448494192\,\delta^2$$

$$+3563965216\,\delta-1823098508=0$$

It possesses the following positive zeros:

$$\delta_1\approx 0.36336, \delta_2\approx 4.22071, \delta_3\approx 889.83069\,.$$

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The minimal deviation of the considered curves equals $\sqrt{\delta_1} \approx 0.60279$. The maximal deviation is attained at the point $X_0 \approx (15.76442, 1.13334)$ in the curve $K_1(x, y) = 0$ and it equals $\sqrt{\delta_2} \approx 2.05443$. This means: point-to-ellipse distance approximation (10) computed at X_0 gives more than 100 % error (Fig.1). Compared with this result, the distance approximation computed by (11) provides the curve $d_{(2)} = 1$ with the shape similar to the previous one but with the lesser maximal deviation, namely ≈ 1.88398 (Fig.2; the ovals of the curves $d_{(1)} = 1$ and $d_{(2)} = 1$ lying inside the ellipse nearly coincide).

Numerical experiments show that the observed growth of the approximation error is typical for the ellipses with small values of the ratio $\min(a, b)/\max(a, b)$.

5 Conclusion

We have treated the problem of finding an analytical approximations for the point-to-ellipse and point-to-ellipsoid distance evaluation problems. Using the analytical representation of the distance values as zeros of appropriate univariate algebraic equations (distance equations), the procedure has been proposed for finding successive approximations for the distance function and for estimation of the approximation errors. For further investigation remains finding a counterpart of Theorem 8 for the distance approximation given by (11) and also the extension of the obtained results to quadrics in multidimensional spaces.

The proposed approach seems to have nice perspectives in application to the problem of establishing the solvability and localization of the set of solutions for a system of multivariate quadric inequalities; the latter arises in Stability and Control Theory [8], [9].

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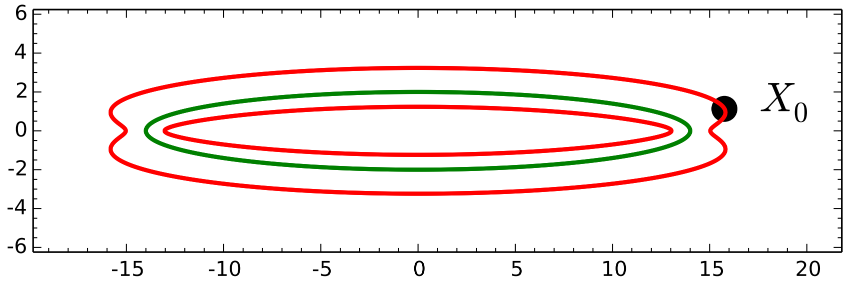

Figure 1: Curve $d_{(1)} = 1$ (in red).

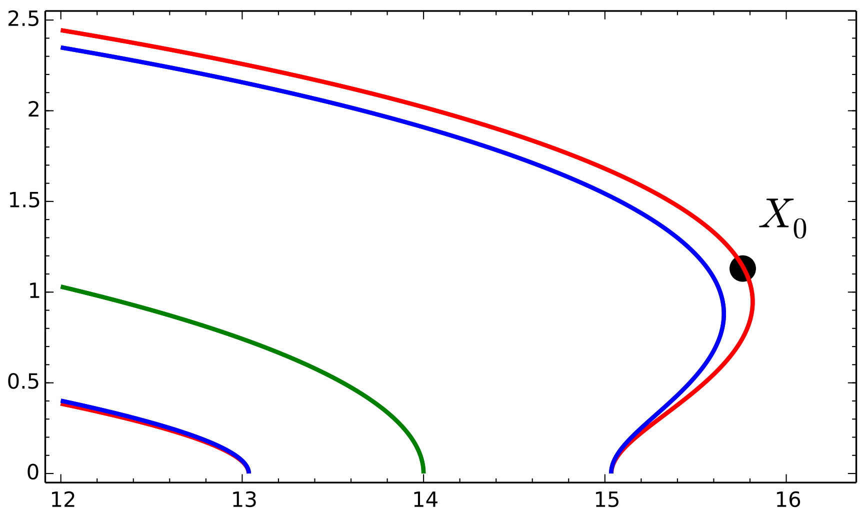

Figure 2: Curve $d_{(2)} = 1$ (in blue) vs. curve $d_{(1)} = 1$ (in red).

