Model description of nonlinear physical processes by the
Lagrangian formalism

S.O. GLADKOV, S.B. BOGDANOVA
Moscow Aviation Institution (National Research University) (MAI)
4 Volokolamskoe shosse, А-80, GSP-3, Moscow 125993
Russia Federation
e-mail: sglad@newmail.ru, sonjaf@list.ru

Abstract: - For an arbitrary scalar parameter \( n(\vec{r},t) \) (density, concentration, temperature, etc.) an invariant with respect to the inversion operation of coordinates and time, most general phenomenological expression for some functional was formed, the role of which, thanks to its main characteristic, is played by generalized Lagrangian function. General nonlinear differential equations are obtained in partial derivatives, and it is shown that in some special cases they can be turned into known nonlinear ordinary differential equations. Depending on the specific parameter values, numerical solutions of certain classes of equations obtained are found, and a wide variety of phase portraits and trajectories are given.

Key-Words: - Lagrangian, functional, nonlinear differential equation, functional derivative.

1 Introduction
Theoretical description of any physical process is associated first of all with the set up of the corresponding physical problem. However, in those cases when it comes to equations of mathematical physics [1 - 12], the main tool is a method for building a particular Lagrange’s function, which is based on the well-known rule of finding the difference between the densities of the kinetic and potential energies. Further, by using the action functional, almost all the basic equations of mathematical physics are obtained, from the Maxwell equations to the Poisson and Laplace equations. The exception here are the equations of parabolic type, which include only the dissipative equations as heat conduction and diffusion equations, and in the special case also the Navier - Stokes equation. To obtain parabolic equations class, artificial method is often used in theoretical physics. It is based on building certain functional \( S \), and then using the phenomenological approach

\[
\frac{\partial n}{\partial t} = \gamma \frac{\delta S}{\delta n},
\]

(1)

(where \( \gamma \) is a certain constant value providing correct equation dimension (1), and symbol \( \frac{\delta}{\delta} \) means drawing of the functional derivative from functional \( S \) ) the unknown equations are found.

2 Problem Formulation
That is why here the main aim is always to build quite specific invariant functional \( S \{n(\vec{r},t)\} \). The problem, which is solved in this paper, is general in nature and reduced to building just the most general form of functional \( S \{n(\vec{r},t)\} \). The only requirement which is applied to subintegral function

\[
L( t,\vec{r},\{n\},\{\bar{n}\},\{\bar{\bar{n}}\},\frac{\partial \{n\}}{\partial x_i},\frac{\partial^2 \{n\}}{\partial x_i \partial x_j},...)
\]

(where \( \{n\} = \{n_\alpha\} = (n_1(\vec{r},t),n_2(\vec{r},t)...n_p(\vec{r},t)) \) - a certain generalized physical parameter (concentration, temperature, density, hydrodynamic flow velocity components etc.), \( t \) - time, \( p \) - quantity of unknown functions, \( \alpha = 1,2,...,p \), indices \( i,k,... = 1,2,3 \) refer to three Cartesian coordinates \( x,y,z \) ) is its invariance with respect to the replacement operations \( t \rightarrow -t, \vec{r} \rightarrow -\vec{r} \).
3 Problem Solution

As a result for the case of two independent functions \( n_0, n_1 \) it can be written that

\[
L \left\{ t, \vec{r}, \{ n \}, \{ \dot{n} \}, \{ \ddot{n} \}, \ldots, \frac{\partial^2 \{ n \}}{\partial x_1 \partial x_1}, \frac{\partial^2 \{ n \}}{\partial x_1 \partial x_2}, \ldots \right\} =
\]

\[
= L (n_0, n_1) = \frac{\alpha}{2} \left( \dot{n}_0^2 + \dot{n}_1^2 \right) - K_0 n_0 n_1 -
\]

\[
\left[ \frac{K_1}{3} \left( n_0^3 + n_1^3 \right) + K_2 \left( n_0 n_1^2 + n_1 n_0^2 \right) \right] +
\]

\[
+ \frac{K_3}{2} \left( n_0^2 + n_1^2 \right) + \frac{\gamma}{2} \left[ \left( \nabla n_0 \right)^2 + \left( \nabla n_1 \right)^2 \right] - \tag{2}
\]

\[
- \gamma_2 \nabla n_0 \nabla n_1 - \frac{\tilde{b}}{2} \left( n_0^2 \nabla n_0 + n_1^2 \nabla n_1 \right) +
\]

\[
+ \frac{\tilde{A}}{3} \left[ \left( \nabla n_0 \right)^3 + \left( \nabla n_1 \right)^3 \right] - \ldots
\]

where all the presented here coefficients \( \alpha, \beta, \gamma, \gamma_2, \lambda, \tilde{b}, \tilde{A}, K_{1,2,3} \) have the relevant dimension, which for the right part of expression (2) by its nature should define the energy density, simply to say the pressure, and \( K_1 = \frac{\beta}{D} \), \( D \) - the diffusion coefficient, \( K_3 = 3K_1 \). It also should be noticed that constant values \( K_0, K_1, K_2 \) and \( K_3 \) will have the meaning of chemical reaction speeds if \( n_0, n_1 \) mean, for example, concentration of the reactive substances in the chemical reaction. In an abstract way, they are just a few functions. Since the action may be specified as (see. references [13-15] as example)

\[
S = \int_{V} \int_{t_0}^{t} L (n_0, n_1) dt \, d^3 x, \tag{3}
\]

then by insertion of (2) into (3) and them into (1), and by making simple calculations of the variation derivatives by the relevant concentrations \( n_a \), the following system of two nonlinear differential equations in partial derivatives can be obtained

\[
\dot{n}_0 = \lambda \left[ K_0 n_1 - K_3 n_0 - \alpha \dot{n}_0 - \tilde{b} \left( n_1 - n_0 \right) \nabla n_0 +
\]

\[
+ \gamma \Delta n_0 + \tilde{A} \nabla \Delta n_0 + K_1 n_1^2 + 2K_2 n_0 n_1 + K_2 n_1^2 \right] \tag{4}
\]

\[
\dot{n}_1 = \lambda \left[ K_0 n_0 - K_3 n_1 - \alpha \dot{n}_1 + \tilde{b} \left( n_1 - n_0 \right) \nabla n_0 +
\]

\[
+ \gamma \Delta n_1 - \tilde{A} \nabla \Delta n_1 + K_1 n_1^2 + 2K_2 n_0 n_1 + K_2 n_0^2 \right] \tag{5}
\]

For the purpose of specificity, we will understand \( n_0, n_1 \) as concentrations of the reacting substances in the chemical reaction. However, it should be emphasized that all the following consideration is general.

Well in view of the just said we introduce the following designation for diffusion coefficient \( D = \gamma \lambda \), and consider the following special cases

1. If \( \alpha = \beta = \tilde{A} = K_0 = K_1 = K_2 = K_3 = 0 \)

as a result we will obtain the regular equation of diffusion

\[
\frac{\partial n_0}{\partial t} = D \Delta n_0. \tag{6}
\]

2. And if \( \frac{1}{\lambda} = \tilde{b} = \tilde{A} = K_0 = K_1 = K_2 = 0 \)

the equation describing the Belousov-Zhabotinskii reaction is obtained.

\[
\dot{n}_0 + \omega_0^2 n_0 = 0, \tag{7}
\]

where frequency is \( \omega_0 = \sqrt{\frac{K_3}{\alpha}} \).

3. \( \frac{1}{\lambda} = \tilde{b} = \tilde{A} = K_0 = K_1 = K_2 = K_3 = 0 \).

As a result the equation of acoustic vibrations (concentration)

\[
\frac{\partial^2 n_0}{\partial t^2} = c_s^2 \Delta n_0, \tag{8}
\]

where sound velocity is \( c_s = \sqrt{\frac{\gamma}{\alpha}} \).

4. \( \alpha = \gamma = K_0 = K_1 = K_2 = K_3 = 0 \).

From equation (4) it appears

\[
\dot{n}_0 = - \lambda \tilde{b} \left( n_1 - n_0 \right) \nabla n_1 + \lambda \tilde{A} \nabla \Delta n_0. \tag{9}
\]
Shall the number of particles in the solution maintains, i.e. \( n_0 + n_1 = \bar{n} = \text{const} \), and the constant vectors \( \bar{b} \) and \( \bar{A} \) have one non-zero component \( \bar{b} = (b,0,0) \), \( \bar{A} = (A,0,0) \), each we can find from here \( \dot{n}_0 = \lambda b (\bar{n} - 2n_0) n'_0 + \lambda A n''_0 \), where the dashes mean differentiation by coordinate \( x \). On the assumption that \( b < 0 \) and \( A > 0 \), we obtain \( \dot{n}_0 + \lambda b (\bar{n} - 2n_0) n'_0 - \lambda A n''_0 = 0 \). If it is necessary to find solution of this equation in the form of a solitary wave by accepting that \( n_0(x,t) = n_0(x - V_0 t) \), where \( V_0 \) is a certain velocity, then we will obtain \( (\bar{b} + V_0) n'_0 - 2\lambda (n_0 - \lambda A n''_0 = 0 \), where dash here means already differentiation by argument \( x - V_0 t \). Finally, after selecting constant \( \lambda \) in form of \( \lambda = \frac{V_0}{3|\bar{b}|\bar{n}} \) in this equation, we immediately get well-known Korteweg-de Vries equation

\[
\dot{n}_0 + \frac{n_0}{\bar{n}} n'_0 + \beta n''_0 = 0, \quad (10)
\]

where constant is \( \beta = \frac{V_0 A}{2|\bar{b}|\bar{n}} \).

So, as we have seen that the system of equations (4) allows us in the relevant special cases obtaining any known differential equations describing a particular physical phenomenon. Our challenge now will be to find possible solutions of the equation system (4-5) by numerical methods in the one-dimensional case, provided that the solution is self-similar form and depends on difference \( x - V_0 t \). We will assume that \( n_0 + n_1 = \bar{n} = \text{const} \). The considering (5), equation (4) can be traced to the following form:

\[
\dot{n}_0 = \frac{V_0}{3|\bar{b}|\bar{n}} [K_0 \bar{n} + 3K_2 \bar{n}^2 - (K_0 + K_3) n_0 + \\
+ (K_1 - K_2) n_0^2 - \alpha (n_0 - c_0^2 \Delta n_0) - \\
- \bar{b} (\bar{n} - 2n_0) \nabla n_0 + \bar{A} \nabla \Delta n_0]...
\]

and in the one-dimensional case it comes from here immediately that

\[
\dot{n}_0 = \frac{V_0}{3|\bar{b}|\bar{n}} [K_0 \bar{n} + 3K_2 \bar{n}^2 - (K_0 + K_3) n_0 + \\
+ (K_1 - K_2) n_0^2 - \alpha (n_0 - c_0^2 \Delta n_0) - \\
- \bar{b} (\bar{n} - 2n_0) \nabla n_0 + \bar{A} \nabla \Delta n_0]...
\]

We will try solution of equation (11) in form of

\[
n_0(x,t) = n_0(x - V_0 t). \quad (11)
\]

After insertion and dimensionalizationless of variables we will have the following nonlinear and nonhomogeneous equation

\[
a_2 y'' - a_3 y' + (1 - a_4) y' - y + \\
+ 2 a_4 y' = a_i
\]

where dimensionless function is \( y = \frac{n_0}{\bar{n}} \), new dimensionless argument is

\[
\xi = \frac{(x - V_0 t)}{K_0 + K_3},
\]

and coefficients are

\[
a_i = \frac{K_0 + 3K_2 \bar{n}}{K_0 + K_3}, a_1 = \frac{K_1 - K_2 ^2 \bar{n}}{K_0 + K_3}, \\
a_2 = \alpha \left( \frac{V_0^2 - c_0^2}{K_0 + K_3} \right), a_4 = \frac{1}{3}, \\
a_3 = \frac{A (K_0 + K_3)^2}{(3|\bar{b}|\bar{n})^2}.
\]

Differentiation in (12) is made by the argument \( \xi \).

If the constant \( b \) is turned to zero then those terms in the equation, which are proportional to the first derivative \( \xi \), disappear and the following nonlinear equation of the third order

\[
a_2 y''' - a_3 y'' - y + a_2 y^2 = -a_i
\]

can be obtained. The above equation, unfortunately, cannot be solved analytically, and the asymptotic cases (for particular values of the parameters, as well as for small values of the argument or function) have very little relevance to the reality. Therefore, in order to study the possible phase trajectories we analyzed a much more complex equation. In some particular cases below its solution is illustrated by various phase portraits and with different values of the parameters. When considering all the other nonlinear terms, which appear in functional (2), equation (12) can be presented in the following rather general form.
By assuming here that \( a_3 = 0 \), we have

\[ y'' = b_0 + b_1 y + b_2 y^2 + b_3 y^3 + \ldots \]  

(15)

To build phase portraits of this equation we should be written in the form of the following system of differential equations

\[
\begin{align*}
    x'_1 &= b_0 + b_1 x_2 + b_2 x_2^2 + b_3 x_2^3 + \ldots, \\
    x'_2 &= x_1.
\end{align*}
\]  

(16)

In Fig. 1-8 some possible phase portraits in coordinates \( x_1 - x_2 \) are illustrated. It is necessary to note that the phase portraits describe the connection of relative concentrations with the speed of their change, therefore not it goes about what acoustic waves here of speech. If we will be interested in the dependence of concentration on the time (this will be shown in the extended version of our paper), these fluctuations will be identical to the acoustic waves.

1. \[
\begin{align*}
    x'_1 &= x_2, \\
    x'_2 &= x_1 + 6x_1^2 - 7x_1^3 - \sin x_2.
\end{align*}
\]  

Fig. 1

2. \[
\begin{align*}
    x'_1 &= x_2, \\
    x'_2 &= -100 \sin x_1 - \sin x_2.
\end{align*}
\]  

Fig. 2

3. \[
\begin{align*}
    x'_1 &= x_2, \\
    x'_2 &= x_1 + 6x_1^2 - 7x_1^3 - \exp x_2.
\end{align*}
\]  

Fig. 3
4. \[
\begin{align*}
    x'_1 &= x_2 \\
    x'_2 &= x_1 + 6x_1^2 - 7x_1^3
\end{align*}
\]

5. \[
\begin{align*}
    x'_1 &= x_2 \\
    x'_2 &= x_1 + x_1^2 - x_1^3 - 7\sin x_2
\end{align*}
\]

6. \[
\begin{align*}
    x'_1 &= x_2 \\
    x'_2 &= 6\sin x_1
\end{align*}
\]

7. \[
\begin{align*}
    x'_1 &= x_2 \\
    x'_2 &= 6\cos x_1
\end{align*}
\]
8. \[
\begin{align*}
x_1' &= x_2 \\
x_2' &= 6 \cos x_1 - 7 \sin x_2
\end{align*}
\]

Fig. 8

**Conclusion**

In conclusion of the paper we should note two main points.

1. A general method for deriving any nonlinear differential equations describing the real physical processes, is suggested;
2. Numerical calculation of the phase portraits for some types of nonlinear differential equations and their graphical illustration are provided.

**References**