Solving a Viral Dynamics Model through Symmetry

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Abstract: We use symmetry analysis to solve the differential equation that govern the dynamics behind viral infection after liver transplants. The traditional regular symmetries usually lead to expressions that are impossible to integrate, subsequently forcing the analyst to consider special cases that may not even be practical. Here we modify the symmetries to avert this.

Key–Words: infectious diseases, viral dynamics, symmetry analysis

1 introduction
There are five known hepatitis viruses. that is, A, B, C, D, and E, or HAV, HBV, HCV, HDV and HEV. The HDV can only propagate in the presence of HBV, hence the need to study both. The model we address here, that we want to solve, is borrowed from Filmann and Herrmann[1], and has the form

\[ \dot{V} = pI - cV, \]
\[ \dot{I} = \beta TV - \delta I, \]
\[ \dot{T} = \lambda - \beta TV - dT. \]

In this model there are three dependent variables, namely \( T, I \) and \( V \). The variable \( T \) represents the size of the uninfected cell population. The variable \( I \) denotes the infected cells, while \( V \) is the free virus particles in serum. It is assumed the uninfected cells are produced at a rate \( \lambda \) and die at the rate \( d \). Uninfected cells \( T \) are assumed to be produced at a constant rate and to die at a rate \( d \). The free virus particles \( V \) are produced at a rate \( p \) proportional to \( I \) and are removed from the system at a rate \( c \). Target cells \( T \) are infected at a rate \( \beta \) proportional to \( TV \). Infected cells \( I \) are killed by the immune system at a rate \( \delta \).

We intend solving the model using Lie’s symmetry group theoretical methods. This technique has difficulties that we remedy through what we call modified symmetry. For the fundamental basics of Lie’s pure approach, the reader is referred to Kallianpur, and Karandikar [2], Kwok [2], Hui [3], Longstaff [4], Platen [5], Naicker, Andriopoulos, and Leach [6], Pooe, Mahomed, and Soh [7], Sinkala, Leach, and OHara [8], Gazizov, and Ibragimov [9].

2 A Lie group symmetrical approach

2.1 The classical approach

Definition 1 Let

\[ \bar{x} = \psi(x; \epsilon) \]

be a family of invertible transformations, of points \( x = (x^1, \ldots, x^N) \in D \subset \mathbb{R}^N \) into points \( \bar{x} = (\bar{x}^1, \ldots, \bar{x}^N) \in R \subset \mathbb{R}^N \), with the parameter \( \epsilon \in S \subset \mathbb{R} \). These are called one-parameter group of point transformations if the following hold.

(i) For each \( \epsilon \in S \), we have the transformations (4) being one-to-one and onto \( D \), meaning \( D \) is not different from \( R \), as \( x^N \) is not different from \( \bar{x}^N \).

(ii) The set \( S \) is a group, say \( G \), with \( \phi(\epsilon, \delta) \) defining the composition law.

(iii) The case \( \bar{x} = x \) corresponds to \( \epsilon = \epsilon_0 \): The identity element of \( G \). That is,

\[ \bar{x}|_{\epsilon=\epsilon_0} = x. \]

or,

\[ \psi(x; \epsilon) \bigg|_{\epsilon=\epsilon_0} = x. \]

(iv) If \( \bar{x} = \psi(x; \epsilon) \) and \( \tilde{x} = \psi(\bar{x}; \delta) \), then

\[ \tilde{x} = \psi(x; \phi(\epsilon, \delta)). \]

We seek here a continuous group of transformations for the equations (1), (2) and (3) through a generator.
subjecting to the condition \( \dot{I} = \beta TV - \delta I \) and it separates into the monomials

\[ \begin{align*}
1 & : \eta^2_t + \dot{I}(\eta^2_I - \xi_t) - \dot{I}^2 \xi_t \\
& - \beta(T \eta^1_I + V \eta^3_I) + \delta \eta^2 = 0, \\
\dot{V} & : \eta^2_V - \dot{I} \xi_V = 0, \\
\dot{T} & : \eta^2_T - \dot{I} \xi_T = 0.
\end{align*} \]  

The third invariance condition gives

\[ \begin{align*}
\eta^3_t + \dot{V} \eta^3_V + \dot{I} \eta^3_I + \dot{T} \eta^3_T - \dot{T}(\xi_t + \dot{V} \xi_V + \dot{I} \xi_I + \dot{T} \xi_T) \\
& + \beta(T \eta^1_I + V \eta^3_I) + d \eta^3 = 0,
\end{align*} \]

subjecting to the condition \( \dot{T} = [\lambda - \beta TV - dT] \) and it separates into the monomials

\[ \begin{align*}
1 & : \eta^3_t + \dot{T}(\eta^3_I - \xi_t) - \dot{T}^2 \xi_t \\
& + \beta(T \eta^1_I + V \eta^3_I) + d \eta^3 = 0, \\
\dot{V} & : \eta^3_V - \dot{T} \xi_V = 0, \\
\dot{I} & : \eta^3_I - \dot{T} \xi_I = 0.
\end{align*} \]

### 2.1.1 Analysis of the monomials

The determining equations lead to the following simplified forms

\[ \xi_V = \xi_I = \xi_T = 0, \]

and

\[ \eta^1_I = 0, \quad \eta^2_I = 0, \quad \eta^1_T = 0, \quad \eta^3_T = 0, \quad \eta^2_V = 0, \quad \eta^2_V = 0, \quad \eta^3_V = 0, \quad \eta^3_V = 0, \quad \eta^3_V = 0. \]

subjecting to the condition \( \dot{V} = pI - cV \) separates into the monomials

\[ \begin{align*}
1 & : \eta^1_I + \dot{V}(\eta^1_V - \xi_t) - \dot{V}^2 \xi_V \\
& - p \eta^2 + c \eta^1 = 0, \\
\dot{I} & : \eta^1_I - \dot{V} \xi_I = 0, \\
\dot{T} & : \eta^1_I - \dot{V} \xi_T = 0.
\end{align*} \]

The second invariance condition gives

\[ \begin{align*}
\eta^2_t + \dot{V} \eta^2_V + I \eta^2_I + T \eta^2_T - \dot{I}(\xi_t + \dot{V} \xi_V + \dot{I} \xi_I + \dot{T} \xi_T) \\
& - \beta(T \eta^1_I + V \eta^3_I) + \delta \eta^2 = 0,
\end{align*} \]

these lead to the symmetries

\[ \begin{align*}
Y_1 & = \frac{\partial}{\partial t}, \\
Y_2 & = e^{\delta t} \frac{\partial}{\partial I}.
\end{align*} \]

There is not much that can be done we these two symmetries in pursuit of solutions. This is why modified symmetries are essential.
2.2 Modified symmetries

Definition 2  Let
\[ \tilde{x} = \chi (\tilde{x}; \omega; \epsilon) \]  \quad (41)

be a family of two-parameters \( \{ \epsilon, \omega \} \subset \mathbb{R} \)
invertible transformations, of points \( \tilde{x} = (\tilde{x}^1(x; \omega; \epsilon), \ldots, \tilde{x}^N(x; \omega; \epsilon)) \in \mathbb{R}^N \) into points \( x = (\tilde{x}^1, \ldots, \tilde{x}^N) \in \mathbb{R}^N \). These we call Neo
one-parameter point transformations when subjected to the conditions
\[ \chi (\tilde{x}; \omega; \epsilon) \bigg|_{\epsilon=0} = \tilde{x}, \]  \quad (42)

and
\[ \tilde{x} |_{\omega=0} = x. \]  \quad (43)

Furthermore,
\[ \chi (\tilde{x}; \omega; \epsilon) \bigg|_{\omega=0} = \tilde{x}, \]  \quad (44)

so that
\[ \chi (\tilde{x}; \omega; \epsilon) \bigg|_{\omega=0, \epsilon=0} = x, \]  \quad (45)

for \( \tilde{x} = (\tilde{x}^1, \ldots, \tilde{x}^N) \in \mathbb{R}^N \) and \( x = (x^1, \ldots, x^N) \in \mathbb{R}^N \).

where \( \tilde{D}_0, \tilde{D}_1, \tilde{A}_0, \tilde{A}_1, \tilde{B}_0, \tilde{B}_1, \tilde{C}_0 \) and \( \tilde{C}_1 \) are constant parameters.

That is,
\[ \eta^1 = i \omega \left( \tilde{A}_0 t + \tilde{A}_1 \right) \cos (i \omega V) + \tilde{D}_0 \cos \left( i \omega \left[ \frac{\tilde{A}_0 t + \tilde{A}_1}{\tilde{D}_0} \right] \right) \sin (i \omega V), \]  \quad (54)

\[ \eta^2 = i \omega \left( \tilde{B}_0 t + \tilde{B}_1 \right) \cos (i \omega I) + \tilde{D}_0 \cos \left( i \omega \left[ \frac{\tilde{B}_0 t + \tilde{B}_1}{\tilde{D}_0} \right] \right) \sin (i \omega I), \]  \quad (55)

\[ \eta^3 = i \omega \left( \tilde{C}_0 t + \tilde{C}_1 \right) \cos (i \omega T) + \tilde{D}_0 \cos \left( i \omega \left[ \frac{\tilde{C}_0 t + \tilde{C}_1}{\tilde{D}_0} \right] \right) \sin (i \omega T), \]  \quad (56)

or simply
\[ \eta^1 = i \omega \left( \tilde{A}_0 t + \tilde{A}_1 \right) \cos (i \omega V) + \tilde{D}_0 \sin (i \omega V), \]  \quad (57)

\[ \eta^2 = i \omega \left( \tilde{B}_0 t + \tilde{B}_1 \right) \cos (i \omega I) + \tilde{D}_0 \sin (i \omega I), \]  \quad (58)

\[ \eta^3 = i \omega \left( \tilde{C}_0 t + \tilde{C}_1 \right) \cos (i \omega T) + \tilde{D}_0 \sin (i \omega T), \]  \quad (59)

These lead to the symmetries
\[ Y_1 = i \omega \frac{\partial}{\partial t} + \sin (i \omega V) \frac{\partial}{\partial V} + \sin (i \omega I) \frac{\partial}{\partial I} + \sin (i \omega T) \frac{\partial}{\partial T}, \]  \quad (60)

\[ Y_2 = i \omega \frac{\partial}{\partial t}, \]  \quad (61)

\[ Y_3 = i \omega \cos (i \omega V) \frac{\partial}{\partial V}, \]  \quad (62)

\[ Y_4 = i \omega t \cos (i \omega V) \frac{\partial}{\partial V}, \]  \quad (63)

\[ Y_5 = i \omega \cos (i \omega I) \frac{\partial}{\partial I}, \]  \quad (64)

\[ Y_6 = i \omega t \cos (i \omega I) \frac{\partial}{\partial I}, \]  \quad (65)

\[ Y_7 = i \omega t \cos (i \omega T) \frac{\partial}{\partial T}, \]  \quad (66)

\[ Y_8 = i \omega \cos (i \omega T) \frac{\partial}{\partial T}. \]  \quad (67)

The argument used in determining the symme-
tries above was the knowledge that an expression of
the form
\[ \xi = a + tb, \]  \quad (68)

can be rewritten in the form
\[ \xi = \frac{a \phi \cos (\omega t / i) + b \sin (\omega t / i)}{\omega / i}. \]  \quad (69)
The latter reduces to the former when $\omega = 0$. For more details on this, the reader is referred to [10] and [11].

### 2.3 Invariance solutions through $Y_1$

Prolongation of $Y_1$:

\[
\begin{align*}
\zeta_1^1 &= \dot{V} \left( \cos (i\omega V) - 1 \right) \quad (70) \\
\zeta_1^2 &= \dot{I} \left( \cos (i\omega I) - 1 \right), \quad (71) \\
\zeta_1^3 &= \dot{T} \left( \cos (i\omega T) - 1 \right). \quad (72)
\end{align*}
\]

\[
\frac{dt}{i\omega t} = \frac{dV}{\sin (i\omega V)} = \frac{d\dot{V}}{V \left( \cos (i\omega V) - 1 \right)} \quad (73)
\]

\[
u_1 = \frac{\tan \left( \frac{i\omega V}{2} \right)}{t}, \quad v_1 = \frac{i\omega \dot{V}}{\left[ \cos (\frac{i\omega V}{2}) \right]^2} \quad (74)
\]

\[
v_1 = \left\{ \frac{\omega^2 \sin \left( \frac{i\omega V}{2} \right) \dot{V}^2}{\left[ \cos (\frac{i\omega V}{2}) \right]^3} + \frac{i\omega \dot{V}}{\cos (\frac{i\omega V}{2})^2} - \frac{\tan \left( \frac{i\omega V}{2} \right)}{t^2} + \frac{i\omega \dot{V} \sec^2 \left( \frac{i\omega V}{2} \right)}{t} \right\} du_1 + C. \quad (76)
\]

This leads to

\[
\lim_{\omega \to 0} \left\{ v_1 - \frac{\omega^2 \sin \left( \frac{i\omega V}{2} \right) \dot{V}^2}{\left[ \cos (\frac{i\omega V}{2}) \right]^3} + \frac{i\omega \dot{V}}{\cos (\frac{i\omega V}{2})^2} - \frac{\tan \left( \frac{i\omega V}{2} \right)}{t^2} + \frac{i\omega \dot{V} \sec^2 \left( \frac{i\omega V}{2} \right)}{t} \right\} = C, \quad (77)
\]

where the parameter $C$ is an integrating constant.

Similarly,

\[
\frac{dt}{i\omega t} = \frac{dI}{\sin (i\omega I)} = \frac{d\dot{I}}{I \left( \cos (i\omega I) - 1 \right)} \quad (78)
\]

\[
u_2 = \frac{\tan \left( \frac{i\omega I}{2} \right)}{t}, \quad v_2 = \frac{i\omega \dot{I}}{\left[ \cos (\frac{i\omega I}{2}) \right]^2} \quad (79)
\]

\[
\lim_{\omega \to 0} \left\{ v_2 - \frac{\omega^2 \sin \left( \frac{i\omega I}{2} \right) \dot{I}^2}{\left[ \cos (\frac{i\omega I}{2}) \right]^3} + \frac{i\omega \dot{I}}{\cos (\frac{i\omega I}{2})^2} - \frac{\tan \left( \frac{i\omega I}{2} \right)}{t^2} + \frac{i\omega \dot{I} \sec^2 \left( \frac{i\omega I}{2} \right)}{t} \right\} = C_2, \quad (80)
\]

\[
dt = \frac{d\dot{T}}{\sin (i\omega T)} = \frac{d\dot{T}}{T \left( \cos (i\omega T) - 1 \right)} \quad (81)
\]

\[
u_3 = \frac{\tan \left( \frac{i\omega T}{2} \right)}{t}, \quad v_3 = \frac{i\omega \dot{T}}{\left[ \cos (\frac{i\omega T}{2}) \right]^2} \quad (82)
\]

\[
\lim_{\omega \to 0} \left\{ v_3 - \frac{\omega^2 \sin \left( \frac{i\omega T}{2} \right) \dot{T}^2}{\left[ \cos (\frac{i\omega T}{2}) \right]^3} + \frac{i\omega \dot{T}}{\cos (\frac{i\omega T}{2})^2} - \frac{\tan \left( \frac{i\omega T}{2} \right)}{t^2} + \frac{i\omega \dot{T} \sec^2 \left( \frac{i\omega T}{2} \right)}{t} \right\} = C_3. \quad (83)
\]

The parameters $C_2$ and $C_3$ are integrating constants.

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**References:**


