Interpolatory Extensions to Univariate Taylor Series: Separate Multinode Ascending Derivative Expansion (SMADE)

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Abstract: This paper has been designed for the presentation of a recently proposed function decomposition method which is called “Separate Multinode Ascending Derivatives Expansion (SMADE)”. The efforts for constructing interpolatory extensions to Taylor series or Taylor decomposition formula which composed of a polynomial and a remainder are not shown first time in this paper. “Separate Node Ascending Derivatives Expansion (SNADE)” has already been proposed by Demiralp and some of his colleagues. SNADE is based on the linear combination of the target function’s derivative values each of which evaluated separate nodes and therefore can be considered as an infinite node interpolatory representation of a univariate function. On the other hand, SMADE linearly combines the all different order derivative values at more than one nodes and hence is an extension to SNADE. Taylor series is based on the derivative values each of which is evaluated at the very same node. Therefore Taylor series can be considered as a very specific case of SNADE such that all nodes merge into a single node by equipping it with a denumarably infinite multiplicity. SNADE is a specific case of SMADE so is the Taylor series. This paper is designed at a conceptual level and hence does not involve any illustrative implementation.

Key–Words: Taylor decomposition formula, Taylor series, Separate node ascending derivatives expansion, Infinite node interpolation.

1 Introduction

Taylor series take an important place in the, univariate or multinomial, function decomposition. It has in fact two parts: (1) a polynomial part for univariate case or a multinomial part for the multivariate case; (2) a remainder part which is a multifold integral over a prescribed order derivative of the function under consideration. The second part is generally called “Remainder” and it can be reduced to a unifold integral certain simple cases. The prescribed order is a positive integer one greater than the polynomial’s degree. If the remainder tends to vanish when this degree grows unboundedly then the two part Taylor formula becomes Taylor series.

The two part Taylor formula can be constructed by using the following identity for a univariate function

\[ f(x) = f(x_e) + \int_{x_e}^{x} d\xi f'\left(\xi\right) \]  

(1)

where \( x_e \) can be called “expansion point”. The validity of this formula needs the differentiability of the target function and its first order derivative’s continuity (and therefore integrability). Since the analytic functions form a quite wide class of functions, the analyticity enables us to comfortably use (1). We will furthermore assume the analyticity of the target function over a region involving its analyticity domain in the complex plane of the independent variable \( x \).

If we replace all appearances of \( f \) in (1) with its first derivative then we can write

\[ f'(x) = f'(x_e) + \int_{x_e}^{x} d\xi f''(\xi) \]  

(2)

which can be combined with (1) to give

\[ f(x) = f'(x_e) + f'(x_e)(x - x_e) + \int_{x_e}^{x} d\xi_1 \int_{x_e}^{\xi_1} d\xi_2 f''(\xi_2). \]  

(3)

where we have used the fact that the integral of 1 between \( x_e \) and \( x \) is equal to \( (x - x_e) \). The integration by parts of the double integral at the right hand allows us to write

\[ f(x) = f(x_e) + f'(x_e)(x - x_e) + \]
If we now define the polynomial and remainder parts and by combining with (1) then we can concisely write

\[ f'(x) = f'(x_e) + f''(x_e)(x - x_e) + \int_{x_e}^{x} d\xi \frac{d}{d\xi} f'''(\xi) \]

(4)

which remains valid without any problem if the target function \( f(x) \) is analytic in a region involving the integration interval. This new formula should also remain when all the appearances of \( f \) are replaced with its first derivative. Hence,

\[ f'(x) = f'(x_e) + f''(x_e)(x - x_e) + \int_{x_e}^{x} d\xi \frac{d}{d\xi} f'''(\xi) \]

(5)

and by combining with (1)

\[ f(x) = f(x_e) + f'(x_e)(x - x_e) + \frac{1}{2} f''(x_e)^2 + \int_{x_e}^{x} d\xi_1 \int_{x_e}^{x_e} d\xi_2 (\xi_1 - \xi_2) f'''(\xi_2) \]

(6)

where the rightmost double integral can be put into more amenable form by using two consecutive integrations by parts and parametric differentiation of an integral. Then we can write

\[ f(x) = f(x_e) + f'(x_e)(x - x_e) + \frac{1}{2} f''(x_e)^2 + \frac{1}{2} \int_{x_e}^{x} d\xi (x - \xi)^2 f'''(\xi). \]

(7)

This is two part Taylor function decomposition formula with second degree polynomial part. We can generalize this formula for the \( n \)th degree polynomial case where \( n \) can be any nonnegative integer

\[ f(x) = \sum_{j=0}^{n} f^{(n)}(x_e)(x - x_e)^j + \frac{1}{n!} \int_{x_e}^{x} d\xi (x - \xi)^n f^{(n+1)}(\xi). \]

(8)

If we now define the polynomial and remainder parts as follows

\[ T_n (x, x_e) = \sum_{j=0}^{n} f^{(n)}(x_e)(x - x_e)^j \]

(9)

\[ R_n (x, x_e) = \frac{1}{n!} \int_{x_e}^{x} d\xi (x - \xi)^n f^{(n+1)}(\xi) \]

(10)

then we can concisely write

\[ f(x) = T_n (x, x_e) + R_n (x, x_e), \]

(11)

where, in last three formulae, the expansion point \((x_e)\) dependence is shown explicitly.

It is not hard to show that the remainder term \( R_n (x, x_e) \) tends to vanish when \( n \) grows unboundedly as long as the target function \( f(x) \) remains analytic in an \(-x\)-complex–plane region involving the integration interval. Then, the formula in (11) turns out to be

\[ f(x) = T_{\infty} (x, x_e) = \sum_{j=0}^{\infty} \frac{1}{j!} f^{(j)}(x_e) (x - x_e)^j \]

(12)

which apparently converges in the analyticity domain of \( f(x) \).

(12) dictates us that the Taylor series is in fact an infinite linear combination of the natural number powers of the term \((x - x_e)\). In other words, these powers form a basis set as long as \( x \) resides in an analyticity disk around \( x_e \) in the complex plane of \( x \). The formulae obtained here contains just a single parameter \( x_e \) where all derivatives of the target function are given. Even though this is a very limited case in the parameterization sense, it is possible to define more than one, and even, denumerably infinite number of parameters. Next sections will involve these cases.

### 2 Separable Node Ascending Derivatives Expansion (SNADE)

Quite recently, Demirralp and his colleagues in his group (Group for Science and Methods of Computing) attempted to develop a new expansion method which uses the target function’s values at different nodes. They called the method “Separate Node Ascending Derivatives Expansion (SNADE)” [1–4]. SNADE uses the target function’s values at different nodes. It is attempted to develop a new expansion method which uses a denumerably infinite locations on the real axis as nodes which could also be considered as complex plane points with complex values. We assume the real-valuedness here for brevity even though it is not hard to extend it to \(-x\)-complex–plane. The starting point for the construction of the method is the following identity again

\[ f(x) = f(x_1) + \int_{x_1}^{x} d\xi f'(\xi) \]

(13)

where \( x_1 \) stands for a real–valued entity (first node) and the analyticity for the target function has been assumed even though it may be relaxed certain cases beyond the analyticity. This formula can be rewritten for a second node denoted by \( x_2 \) however this time for the first derivative of the target function to get

\[ f'(\xi) = f'(x_2) + \int_{x_2}^{\xi} d\xi_1 f''(\xi_1) \]

(14)

which can be combined with (14) to give

\[ f(x) = f(x_1) + f'(x_2)(x - x_1) \]
This somewhat nested identity can be generalized to the following equality by using the integral–of–derivative identity for the second derivative at the third node symbolized by $x_3$ and then by combining the result with (15)

$$f(x) = f(x_1) + f'(x_2)(x - x_1) + \frac{1}{2}(x - x_1)(x + x_1 - 2x_2) f''(x_3) + \int_{x_1}^{x} d\xi_1 \int_{x_2}^{\xi_1} d\xi_2 f'''(\xi_2).$$

(15)

If we define the following entities

$$\phi_j(x, x_1, ..., x_j) \equiv I_j(x_1, ..., x_j)1_f,$$

$$j = 0, 1, ...$$

(22)

then we can write the following integral recursion

$$\phi_i(x; x_1, ..., x_i) = \int_{x_1}^{x} d\xi_1 \phi_{i-1}(\xi_1; x_2, ..., x_i),$$

$$i = 1, 2, ...$$

(23)

whose differentiation with respect to $x$ at both sides gives

$$\phi_i'(x; x_1, ..., x_i) = \phi_{i-1}'(x; x_2, ..., x_i),$$

$$i = 1, 2, ...$$

(24)

If we specifically choose the case where $j = i$ then we can conclude by solving the obtained first order recursion

$$\phi_i^{(i)}(x; x_1, ..., x_i) = 1,$$

$$i = 1, 2, ...$$

(26)

If we now specifically take $x = x_j$ in (25) then we can proceed through the following equalities

$$\phi_i^{(j)} \left( \begin{array}{c} x_j; x_1, ..., x_j \ 
 i \end{array} \right) = \phi_{i-1}^{(j-1)} \left( \begin{array}{c} x_j; x_2, ..., x_j \ 
i-1 \end{array} \right)$$

$$= \phi_{i}^{(j-2)} \left( \begin{array}{c} x_j; x_3, ..., x_j \ 
i-2 \end{array} \right)$$

$$= \phi_{i-1}^{(1)} \left( \begin{array}{c} x_j; x_j, ..., x_j \ 
i-j+1 \end{array} \right)$$

$$= 0$$

$$i = 1, 2, 3, ...; \quad j = 1, 2, ..., i - 1$$

(27)

As can be noticed immediately these polynomials’ all derivatives except the one whose order is equal to the degree of the polynomial vanish at the relevant node. The case where derivative does not vanish the polynomial takes the value of 1. This is somehow an expected behavior from the $\phi$ polynomials. However, beyond that, this shows that the $\phi$ polynomials have interpolative natures since $\phi_j$ picks the value of the
target function’s $j$th derivative at the point where $x$ equals to $x_{j+1}$. The very specific case where all nodes are located at the very same position say the point denoted by $x_e$ enables us to write

$$\phi_i(x; x_e) \equiv \frac{(x - x_e)^i}{i!}, \quad i = 0, 1, 2, \ldots$$

(28)

and therefore

$$\phi^{(j)}_i(x_e; x_e) = \delta_{i,j}, \quad i, j = 0, 1, 2, \ldots$$

(29)

All these mean that the Taylor series is a very specific case of SNADE where all nodes become just a single node with denumerably infinite multiplicity.

SNADE is currently under an intense study in Demiralp’s group. There seem to be a lot of interesting openings at the horizon. We find this information here sufficient for our purposes.

### 3 Separate Multinode Ascending Derivative Expansion (SMADE)

SNADE is based on an interpolative approach where each derivative of the target function is evaluated at a single and different node even though each of which may have certain same values in very specific cases. SNADE has been considered as an extension to Taylor series even though further extensions can be added even to SNADE. The author of this paper quite recently considered the cases where each derivative of the target function is associated by not a single node but a set of separate nodes whose numbers may change from derivative to derivative. He called the resulting method “Separate Multinode Ascending Derivative Expansion (SMADE)”. Even though this methods finds its basic in most recent private communications this paper is the first official announcement of the issue.

SMADE is based on the following set of identities

$$f(x) = f(x_i) + \int_{x_i}^{x} \frac{d^j f}{d\xi^j}(\xi) \, d\xi, \quad i = 1, 2, \ldots, m$$

(30)

which are in fact the original integral of derivative identity for $m$ number of different nodes. We need to combine these equations through certain unknown entities such that the resulting single identity gives the $j$th of the identities given by (30) when the independent variable $x$ becomes $x_j$. To this end we can take $m$ number linearly independent function of $x$, which are denoted by subindexed $\alpha(x)$s such that

$$\sum_{i=1}^{m} \alpha_i(x) = 1.$$  

(31)

If we now multiply the both sides of (30) by $\alpha_i(x)$ and then take the sums of both sides between 1 and $m$ inclusive and use (31) we get

$$f(x) = \sum_{i=1}^{m} \alpha_i(x) f(x_i) + \sum_{i=1}^{m} \alpha_i(x) \int_{x_i}^{x} \frac{d^j f}{d\xi^j}(\xi) \, d\xi,$$

(32)

This equation becomes giving the values of the target function values at the nodal point values of the independent variable $x$ if the following equations are satisfied

$$\alpha_i(x_j) = \delta_{i,j}, \quad i, j = 1, 2, \ldots, m.$$  

(33)

This urges us to write the following structural equation

$$\alpha_i(x) = L_i^{(m-1)}(x), \quad i = 1, 2, \ldots, m$$

(34)

where $L_i^{(m-1)}(x)$ stands for the Lagrange intepolation polynomials whose explicit structures are given below

$$L_i^{(m-1)}(x) \equiv \prod_{j=1}^{m} \frac{(x - x_j)}{(x_i - x_j)}, \quad i = 1, 2, \ldots, m$$

(35)

where the superscript $(m - 1)$ denotes the degree of the Lagrange polynomial under consideration. The Lagrange polynomials satisfy the following equalities

$$L_i^{(m-1)} (x_j) = \delta_{i,j}, \quad i, j = 1, 2, \ldots, m$$

(36)

which has been the reason why we have chosen $\alpha(x)$ functions as Lagrange polynomials.

The Lagrange interpolation of a given function $f(x)$ can be written as follows

$$f(x) \approx \sum_{i=1}^{m} f(x_i) L_i^{(m-1)}(x)$$

(37)

which turns out to be an exact equation when the function $f(x)$ becomes a polynomial with a degree not exceeding $(m - 1)$. Among these equations the following one is perhaps the most important one

$$\sum_{i=1}^{m} L_i^{(m-1)}(x) = 1$$

(38)

which confirms the fact that the sum of $\alpha(x)$ functions over the integer subindices from 1 to $m$ inclusive is just 1.

We have been sufficiently equipped to rigorously construct SMADE now. We are going to start with the
construction of two part decomposition, one for polynomial part and one for remainder term. We are going to start with the least order terms first. Because of the multinodal structure the degree of the polynomial part will not be 0 for this time. It will be a higher degree (equal to the number of nodes minus 1) polynomial. Towards this goal we can consider \( m_0 \) number of separate nodes superscripted by \((0)\) beside the ordering subindices. In this case the above analysis let us to write

\[
f(x) = \sum_{i=1}^{m_0} f(x_i^{(0)}) L_i^{(m_0-1)}(x, x^{(0)}) + \sum_{i=1}^{m_0} L_i^{(m_0-1)}(x, x^{(0)}) \int_{x_i^{(0)}}^{x} d\xi f' (\xi)
\]

where \( x^{(0)} \) stands for the collection of \( m \) number of \( x_i^{(0)} \)’s subindexed between 1 and \( m \) inclusive.

The next step is same as the one in (39) except the replacement of every appearances of \( f \) with its first derivative and the replacement of the nodes with the ones (whose numbers now may change) with a superscript \((1)\). Hence we can write

\[
f'(x) = \sum_{i=1}^{m_1} f'(x_i^{(1)}) L_i^{(m_1-1)}(x, x^{(1)}) + \sum_{i=1}^{m_1} L_i^{(m_1-1)}(x, x^{(1)}) \int_{x_i^{(1)}}^{x} d\xi f' (\xi)
\]

where \( x^{(1)} \) denotes the collection of the \( x_i^{(1)} \) nodal values indexed between 1 and \( m_1 \) inclusive.

Now we can combine (40) with (39) and get

\[
f(x) = \sum_{i=0}^{m_0} f(x_i^{(0)}) L_i^{(m_0-1)}(x, x^{(0)}) + \sum_{i=1}^{m_1} f(x_i^{(1)}) \left( \sum_{i=1}^{m_0} L_i^{(m_0-1)}(x, x^{(0)}) \right)
\]

\[
\times \int_{x_i^{(0)}}^{x} d\xi L_i^{(m_1-1)}(\xi, x^{(1)})
\]

\[
+ \text{Remainder}
\]

which can be concisely rewritten as follows

\[
f(x) = \sum_{i_0=1}^{m_0} f(x_i^{(0)}) \phi_{i_0}^{(0)} (x; x^{(0)}) + \sum_{i_1=1}^{m_1} f(x_i^{(1)}) \phi_{i_1}^{(1)} (x; x^{(1)}) + \text{Remainder}
\]

where

\[
\phi_{i_0}^{(0)} (x; x^{(0)}) \equiv L_i^{(m_0-1)}(x),
\]

\[
\phi_{i_1}^{(1)} (x; x^{(1)}) \equiv \sum_{i_0=1}^{m_0} L_i^{(m_0-1)}(x) \int_{x_i^{(0)}}^{x} d\xi L_i^{(m_1-1)}(\xi, x^{(1)})
\]

\[
\times L_i^{(m_1-1)}(\xi_1)
\]

These results can be generalized to any finite order for two part decomposition formula as follows

\[
f^{(j)}(x) = \sum_{i=1}^{m_1} f^{(j)}(x_i^{(j)}) L_i^{(m_j-1)}(x, x^{(m_j)}) + \sum_{i=1}^{m_1} L_i^{(m_j-1)}(x, x^{(m_j)}) \int_{x_i^{(m_j)}}^{x} d\xi f^{(j+1)} (\xi),
\]

\[
j = 0, 1, 2, ...
\]

which enable us to write the following \( n \)th order two part formula

\[
f(x) = \sum_{j=0}^{n} \sum_{i_j=1}^{m_j} f(x_i^{(j)}) \phi_{i_j}^{(j)} (x; x^{(j)}) + \text{Remainder}_n
\]

where the basis functions \( \phi_{i_j}^{(j)} (x; x^{(j)}) \)’s are defined accordingly as we previously did. The series representation for SMADE can be written as follows

\[
f(x) = \sum_{j=0}^{\infty} \sum_{i_j=1}^{m_j} f(x_i^{(j)}) \phi_{i_j}^{(j)} (x; x^{(j)})
\]

if the \( n \)th order remainder term tends to vanish when \( n \) grows unboundedly. These finalize the construction of SMADE even though there remains a lot of important properties to be investigated. The studies in Demiralp’s group are focused now some of them as we expect to report the findings starting soon.

4 Concluding Remarks

This work has been devoted to the presentation of a new interpolative method which can be considered an extension to Taylor series representation or to two part Taylor function decomposition formula where the first part is a polynomial while the second part corresponds to remainder term under integration. This is not the first attempt for making an interpolative extension to Taylor series. The author and his colleagues in his group have studied and are continuing to study on SNADE which is an acronym for “Separate Node Ascending Derivatives Expansion”. In both Taylor series and SNADE series the \( n \)th additive term is a polynomial of \( n \)th degree, while in the method presented...
here and we call SMADE which an acronym for “Separate Multinode Ascending Derivatives Expansion”, it is again a polynomial but with a degree which is pretty much higher than $n$.

In this work, we have emphasized on Taylor series conceptualism based on a well-known integral of derivative identity and at the end the infinite node interpolatory nature. SNADE has also been recalled up to certain level of detailing. SMADE has also similar natures. The distinguishing properties of these three methods are as follows: (i) All these methods use a denumerable infinite number of nodes even though the Taylor series uses just a single node, the expansion point, but with denumerably infinite multiplicity; (ii) SNADE can use a denumerably infinite number of separate nodes such that each derivative of the target function is evaluated at a different node although certain infinite or finite number of nodes may have same values; (iii) In SMADE each derivative enters the formula with the values evaluated at finite number of separate nodes.

In both SNADE and SMADE there are many action waiting issues for future studies some of which have already started in Demiralp’s group.

References: