Extremal Curves of a Total Curvature Type Energy

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Abstract: We analyze the variational problem associated to a total curvature type energy when acting on suitable spaces of curves in Riemannian manifolds with constant sectional curvatures, paying special attention to closed critical curves. According to a recent mathematical model for the primary visual cortex V1, minimizers of this energy are linked to subriemannian geodesics in the unit tangent bundle of $\mathbb{R}^2$. A few numerical experiments concerning these geodesic are made by using a numerical approach which is based on a gradient descent method. The experiments are extended to the case in which the determining functional is the elastic energy.

Key–Words: elasticae, total curvature, extremal curves, real space forms, unit tangent bundle, subriemannian manifold.

1 Introduction

Regular curves in Riemannian n-manifolds M^n which are critical points of the elastic energy $\int_\gamma \kappa^2$ (κ being the curvature of the curve γ) have been intensively investigated under various points of view. Extremals of this energy correspond to the model for classical *elasticae* proposed by D. Bernoulli around 1740 which have been widely studied (see for instance, [12, 20, 21, 22, 23]). If the Riemannian manifold is $\mathbb{R}^3$ they can be used also to model stiff rods, stiff polymers, vortices in fluids, superconductors, membranes and mechanical properties of DNA molecules (for more details see [7, 28] and the references therein). If $M^n(\rho)$ is a real space form of constant sectional curvature ρ it can be proved that elastic curves must lie in a totally geodesic 3-dimensional submanifold of $M^n(\rho)$. In particular, if $M^n(\rho) = \mathbb{R}^2$ their possible shapes where discovered by L. Euler. There are no closed plane free elasticae. Closed classical elasticae in 2-dimensional round spheres and in the hyperbolic plane have been classified in [21]. Using a Lagrange multiplier argument, extremals of the bending energy among curves with the same length can be considered as critical curves of $\int_\gamma \kappa^2 + \lambda$, $\lambda \in \mathbb{R}$. Only two types of closed extremals in $\mathbb{R}^2$ appear in this case: circles and Bernoulli's eight figure. Moreover, closed extremals of $\int_\gamma \kappa^2$ in $\mathbb{R}^2$ among curves with same length and enclosed area have applications in material science [29]. Closed classical elastic curves in $\mathbb{R}^3$ and in the 3-sphere $\mathbb{S}^3$ have been analyzed in [22].

On the other hand, curve optimization plays a major role in imaging and visual perception. Mostly these optimal curve models rely on Euler's elastica, but this approach has two important problems; first, there are many local minimizers which are not global and, while local stationarity can be reasonably checked, global optimality is much more difficult to deal with; and second, the boundary value problem for elastica is very hard to solve analytically. To overcome these problems another approach based on subriemannian geodesics has been raised recently [11, 17]. A version of the subriemannian geodesic approach is explained in the following section and leads to minimization of $\int_\gamma (\kappa^2 + a^2)^{\frac{1}{2}}$ in $\mathbb{R}^2$. Thus a natural extension is to consider the variational problem associated to the energy defined on suitable spaces of immersed curves in Riemannian manifolds by

$$\mathcal{F}_a(\gamma) = \int_\gamma \left(\kappa^2 + a^2\right)^{\frac{1}{2}} ds, \tag{1}$$

where, $a \in \mathbb{R}$, s is the arclength parameter and $\kappa(s)$ is the geodesic curvature of $\gamma(s)$.

If $a = 0$, then $\mathcal{F}_a$ is nothing but the *total curvature functional* which has been associated to certain models of massless particles with rigidity [27]. From the mathematical point of view, this is a trivial variational problem in $\mathbb{R}^2$ due to the classical Whitney and Grauestein's result. Moreover, as a consequence of another classical result due to Fenchel, the minimum of the total curvature action over simple closed curves in the Euclidean 3-space is 2π and it is reached

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precisely on convex plane curves. In a more general setting, one may want to consider the total curvature functional acting on suitable spaces of curves of a Riemannian space, and then to study the associated variational problem. Although the study of the total curvature of curves in Riemannian spaces has been intensively considered along the literature (see [16] and references therein), the systematic study of the associated variational approach was initiated in [2, 3]. In fact, the variational approach for the total curvature in surfaces was first considered in [2], where it was shown that extremals of the total curvature are reached by curves consisting of parabolic points. Stability of extremals was also holographically characterized there. Then, the problem in spaces with the highest rigidity (constant curvature) was considered in [3], where it was proved that the dynamics associated with the total curvature action is consistent only in round 3-spheres. More precisely, it is shown in [3] that extremals of the total curvature in an n-dimensional Riemannian space, $M^n(\rho)$, with constant curvature ρ, must lie in a totally geodesic submanifold $M^m(\rho) \subset M^n(\rho)$, $m \leq 3$ and, in addition, we must have $\rho \geq 0$. So it does not make sense in, for example, the hyperbolic space. Moreover, if $\rho = 0$ and up to topology, arbitrary plane curves in $\mathbb{R}^3$ are the critical curves. Finally, for $\rho > 0$, up to topology, extremals over a round 3-sphere are just the so called *Legendrian curves*, that is, curves which are horizontal lifts, via the Hopf map, of curves in the 2-sphere. As in the elasticae case, one may consider extremals among curves of the same length and/or same total torsion. Again, by a Lagrange multiplier argument, these may be treated as critical curves for $\int_\gamma (m + n\kappa + p\tau)$, $m, n, p \in \mathbb{R}$ and τ representing the torsion of γ. These curves have been used as models for relativistic particles in pseudo-riemannian ambient spaces and if $p = 0$ they were proposed as a varitional model to described protein chains in [18]. In this case, the family of extremals is formed by Lancret helices. There are no closed Lancret helices in $\mathbb{R}^3$ (other that plane curves) but there exist a rational 1-parameter family of closed extremals in $\mathbb{S}^3$ (see [1] and references therein for more details). Other partial results, providing examples and families of extremals in Berger spheres and in the complex projective plane $\mathbb{CP}^2(4)$, can be found in [3, 10] (see also [8, 13, 15] for extremals in warped product spaces). The existence of extremals for the total curvature of curves in homogeneous 3-spaces M^3 has been recently studied in [9].

Here we consider the variational problem associated to (1), for $a \neq 0$, when acting on spaces of curves, satisfying suitable boundary conditions, of a riemannian n-manifold M^n. As it has been said before, if $a \neq 0$, planar extremals are relevant in image restoring and a parametrization of them without inflection points has been given in [11]. This connection is described in section 2. Then, in section 3, we will compute the Euler-Lagrange equation for (1) and we will show that a simple analysis of the Euler-Lagrange's first integrals phase plane reveals that closed extremals in background spaces with constant sectional curvature ρ, $M^n(\rho)$, are only possible if $\rho \geq 0$. Moreover, in this case, that is when the ambient space is a two sphere $M^2(\rho) = \mathbb{S}^2(\rho)$, we have that if $a^2 > \rho$ the only closed extremals are geodesics, while if $a^2 = \rho$, every circle (and only them) is a closed critical curve. If $a^2 < \rho$, there is a 1-parameter family of closed extremals with non-constant curvature, [**?**]. Also, a 2-parameter family of closed helical extremals can be found in $\mathbb{S}^3(\rho)$, [4]. In this section we also characterize equivariant surfaces all whose orbits are critical for (1) and show that they must be either cylinders or surfaces with positive constant curvature. The final section is devoted to analyze numerically the subriemannian geodesic problem in the unit tangent bundle $\mathbb{R}^2 \times \mathbb{S}^1$. This analysis is made by using the XEL-platform implemented in [5] and we illustrate how it could be extended to many other similar functionals under different boundary conditions as, for example, the elastic energy of curves.

2 Subriemannian Geodesics in $\mathbb{R}^2 \times \mathbb{S}^1$

Neuro-biologic research over the past few decades has greatly clarified the functional mechanisms of the first layer V1 of the visual cortex (*primary visual cortex*). Such layer contains a variety of types of cells, including the so-called simple cells. Researchers found that V1 constitutes of orientation selective cells at all orientations for all retinal positions so simple cells are sensitive to orientation specific brightness gradients (for details see [11, 17]).

Recently, this structure of the primary visual cortex has been modeled using subriemannian geometry, [26]. In particular, the unit tangent bundle of the plane can be used as an abstraction to study the organization and mechanisms of V1.

According to this model, in the space $\mathbb{R}^2 \times \mathbb{S}^1$ each point (x, y, θ) represents a column of cells associated to a point of retinal data $(x, y) \in \mathbb{R}^2$, all of which are adjusted to the orientation given by the angle $\theta \in \mathbb{S}^1$. In other words, the vector $(\cos\theta, \sin\theta)$ is the direction of maximal rate of change of brightness at point (x, y) of the picture seen by the eye. Such vector can be seen

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as the normal to the boundary of the picture. Thus, when the cortex cells are stimulated by an image, the border of the image gives a curve inside the $3D$ space $\mathbb{R}^2 \times \mathbb{S}^1$, but such curves are restricted to be tangent to the distribution spanned by the vector fields

$$X_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \quad , \quad X_2 = \frac{\partial}{\partial \theta} . \qquad (2)$$

It is believed that, if a piece of the contour of a picture is missing to the eye vision (or maybe it is covered by an object), then the brain tends to "complete" the curve by minimizing some kind of energy, being length the simplest (but not the only) of such. In short, there is some subriemannian structure on the space of visual cells and the brain considers a subriemannian geodesic between the endpoints of the missing data.

Let M^n be a smooth manifold. A a sub-bundle of the tangent bundle TM is called *distribution D* on M^n. Once we have chosen D, a *D-curve* on M^n is a smooth immersed curve $\gamma : [a, b] \to M$ which is always tangent to D, that is, $\gamma'(t) \in D_{\gamma(t)}$ for all $t \in [a, b]$. A distribution D is said to be *bracket generating* if for every $p \in M$ the sections of D near p together with all their commutators span the tangent space of M^n at p, T_pM. By a well known theorem of Chow, there is a D-curve joining any two points of M^n if D bracket generating (check [10] for the smooth version of this theorem). A *subriemannian metric* is a smoothly varying positive definite inner product $\langle , \rangle$ on D. Thus, if D were equal to the whole tangent bundle, $\langle , \rangle$ would give a Riemannian metric on M^n. A *subriemannian manifold*, $(M, D, \langle , \rangle)$, is a smooth n-dimensional manifold M^n equipped with a subriemannian metric $\langle , \rangle$ on a bracket generating distribution D of rank $m > 0$. In this case, the *length* of a D-curve $\gamma : [a, b] \to M$ is defined to be $L(\gamma) = \int_a^b \langle \gamma'(t), \gamma'(t) \rangle)\, dt$. Since D is bracket generating it is possible to endow M^n with a *distance d*. The distance $d(p, q)$ between any two points p and q of M^n is defined by $d(p, q) = inf_\gamma\{L(\gamma)/\gamma \,\text{is a}\, D - \text{curve joining}\, p \,\text{to}\, q\}$.

To construct a subriemannian structure on $M^n = \mathbb{R}^2 \times \mathbb{S}^1$ we take the distribution $D = ker(\sin\theta\, dx - \cos\theta\, dy)$, where x and y are the coordinates on $\mathbb{R}^2$ and θ is the coordinate on $\mathbb{S}^1$. This distribution is spanned by the vector fields described in (2). Consider on D the inner product $\langle , \rangle$ for which the two vectors (2) are everywhere orthonormal. Every D-curve $\gamma(t) = (x(t), y(t), \theta(t))$ with $\gamma * (\sin\theta\, dx + \cos\theta\, dy) \neq 0$ is the lift of a regular curve $\alpha(t) = (x(t), y(t))$ in the plane whose tangent vector $\alpha'(t)$ forms the angle $\theta(t)$ with the x-axis, i.e.,

$$\alpha'(t) = v(t)\, \cos\theta \frac{\partial}{\partial x} + v(t)\, \sin\theta \frac{\partial}{\partial y} , \qquad (3)$$

where $v(t)$ is the speed of $\alpha(t)$. Conversely, every regular curve $\alpha(t)$ in the plane may be lifted to a D-curve $\gamma(t) = (x(t), y(t), \theta(t))$ by setting $\theta(t)$ equal to the angle between $\alpha'(t)$ and the x-axis. Now, the tangent vector $\gamma'(t)$ of the D-curve $\gamma(t)$ has squared length

$$\begin{aligned} \langle \gamma'(t), \gamma'(t) \rangle &= v^2(t) + \theta'^2(t) \\ &= v^2(t) \left(1 + \left(\frac{\theta'(t)}{v(t)} \right)^2 \right) \\ &= v^2(t)(1 + \kappa^2(t)) , \end{aligned} \qquad (4)$$

where $\kappa(t)$ is the curvature of α, and so the length of $\gamma(t)$ is equal to the integral of $\sqrt{1 + \kappa^2(t)}v(t)$ along α. Thus the D-curves with $(\sin\theta\, dx + \cos\theta\, dy) \neq 0$ that realize the distance between two points (x_0, y_0, θ_0) and (x_1, y_1, θ_1) of M^n are the lifts of curves α in the plane joining (x_0, y_0) to (x_1, y_1) with initial angle θ_0 and final angle θ_1 that minimize the functional

$$L(\kappa) = \int \left(1 + \kappa^2(s)\right)^{\frac{1}{2}} \, ds , \qquad (5)$$

s being the arc-lenth parameter, among all such curves in the plane. In other words, geodesics in V1 are obtained by lifting to $M^n = \mathbb{R}^2 \times \mathbb{S}^1$ minimizers of (5) in $\mathbb{R}^2$.

Finally, as indicated in [11], the hypercolumnar organization of the visual cortex suggests that the cost of moving one orientation unit is not necessarily the same as to moving spacial units, then the curve completion problem should consider the functional $\mathcal{F}_a$ acting on planar curves instead. This motivates considering critical curves of $\mathcal{F}_a$, not only in the plane, but also in more general backgrounds.

3 Extremals in Real Space Forms

For finite m, let M^m be an m-dimensional Riemannian manifold with metric $\langle \, , \, \rangle$ and associated Levi-Civita covariant derivative ∇. Let $\beta(t)$, $\beta : \mathbb{I} := [0, 1] \to M$ be a C^∞ immersed curve and denote by $\gamma(s)$ its unit speed reparametrization, that is, it satisfies $\langle \gamma'(s), \gamma'(s) \rangle \equiv 1$, where now $'$ denotes derivative with respect to the arc-length parameter $s \in [s_0, s_1]$. If the successive covariant derivatives of the velocity vector

$$\gamma'(s),\ \nabla_{\frac{\partial}{\partial s}} \gamma'(s),\ \nabla^2_{\frac{\partial}{\partial s}} \gamma'(s), \ldots, \nabla^{n-1}_{\frac{\partial}{\partial s}} \gamma'(s),$$

are everywhere linearly independent, for $1 \leq n \leq m$, then we set $e_0 := \gamma'(s)$, and define the *unit normal* *field* e_1 to be the unit vector field along γ in the direction of $\nabla_{\frac{\partial}{\partial s}}\gamma'(s)$. The *geodesic curvature* is defined by

$$\kappa_1(s) \;:=\; \langle \nabla_{\frac{\partial}{\partial s}} e_0(s), e_1(s)\rangle.$$

Unit normal fields e_j and curvatures κ_j, for $j = 2, \ldots, n-1$, are given inductively by Gramm-Schmidt orthogonalization, as follows. Let $\hat{e}_j(s)$ be the orthogonal projection of $\nabla_{\frac{\partial}{\partial s}} e_{j-1}$ onto the orthogonal complement of $\{e_0(s), e_1(s), e_2(s), \ldots, e_{j-1}(s)\}$. Set

$$e_j(s) \;:=\; \frac{\hat{e}_j(s)}{\|\hat{e}_j(s)\|}\,,$$

and $\kappa_j(s) \;:=\; \langle \nabla_{\frac{\partial}{\partial s}} e_{j-1}, \hat{e}_j(s)\rangle$. Then κ_j is never-zero, and the *Frenet-Serret* formulae read as

$$\begin{aligned} \nabla_{\frac{\partial}{\partial s}} e_{j-1} &= \kappa_j(t)e_j(t) - \kappa_{j-1}(t)e_{j-1}(t)\,, && (6)\\ \nabla_{\frac{\partial}{\partial s}} e_{n-1} &= -\,\kappa_{n-1}(t)e_{n-1}(t). && (7) \end{aligned}$$

We call *rank* of γ the maximum integer j for which $\kappa_j(s) \neq 0$ for some $s \in [s_0, s_1]$, namely, $\operatorname{rank}(x) = n-1$ where n is the maximum integer for which $\gamma'(s),\ \nabla_{\frac{\partial}{\partial s}}\gamma'(s), \ldots, \nabla^{n-1}_{\frac{\partial}{\partial s}}\gamma'(s)$, are linearly independent. So $0 \le \operatorname{rank}(\gamma) \le m-1$, and γ is a geodesic if and only if $\operatorname{rank}(\gamma) = 0$. When γ has constant curvatures κ_l for $1 \le l \le \operatorname{rank}(\gamma)$, it is called a *helix* of rank l.

If $M^m(\rho)$ is a Riemannian n-manifold with constant sectional curvature ρ, then curves of any rank are determined by their curvatures up to isometries and, if the rank of γ is $n-1$, with $2 \le n < m$, then γ is contained into an n-dimensional totally geodesic submanifold, $M^n(\rho)$ of $M^m(\rho)$.

In the first part of this section we will often resort to results in [4] (which were proved inspired by techniques in [21] and [22]). Although in this section we will be mainly concerned with closed curves, many computations can be performed in larger spaces of curves, so let us first consider the space Ω of immersed curves $\gamma(t)$ in a m-dimensional Riemannian manifold M^m with fixed endpoints $p, q \in M^m$: $\Omega = \{\gamma\colon [0,1] \to M^m;\ \gamma \text{ is a } C^\infty\text{-immersion};\ \gamma(0) = p; \gamma(1) = q\}$. Let us define the following energy functional $\mathcal{F}_a\colon \Omega \to \mathbb{R}$

$$\mathcal{F}_a(\gamma) = \int_\gamma \left(\kappa^2 + a^2\right)^{\frac{1}{2}}\, ds, \qquad (8)$$

where, $a \in \mathbb{R}$, s is the arc-length parameter and $\kappa(s) \equiv \kappa_1(s)$ is the geodesic curvature of $\gamma(s)$. Observe that geodesics (curves of rank 0) are minima of $\mathcal{F}_a$ among curves of the same length.

For a given $\gamma \in \Omega$, we consider a C^∞ variation by curves in Ω, that is a C^∞ function $\Gamma\colon (-\epsilon, \epsilon) \times [0,1] \to M^m$ such that $\Gamma(0,s) = \gamma(s)$ and $\Gamma(z,s) = \gamma_z(s) \in \Omega$. We say that $\gamma(s)$ is a *critical curve* or, simply, an *extremal* of $\mathcal{F}_a$ in Ω, if $\frac{\mathrm{d}\,\mathcal{F}_a}{\mathrm{d}\,z}(0) = 0$, for any variation of γ. By using standard arguments one can compute the first variation formula of $\mathcal{F}_a$

$$\frac{\mathrm{d}\,\mathcal{F}_a}{\mathrm{d}\,z}(0) = \int_\gamma \langle \mathcal{E}(\gamma), W\rangle ds + \mathcal{B}[\gamma, W]\,, \qquad (9)$$

where $W = \frac{\partial \Gamma}{\partial z}(0,s)$ is the variation field of Γ and $\mathcal{E}(\gamma)$, $\mathcal{B}[\gamma, W]$ are the Euler-Lagrange and boundary operators, respectively. After a long computation, we obtain that the Euler-Lagrange operator is given by

$$\begin{aligned} \mathcal{E}(\gamma) = {} & \frac{1}{(\kappa^2+a^2)^{\frac{1}{2}}}\nabla_T^3 T + 2\frac{d}{ds}\left(\frac{1}{(\kappa^2+a^2)^{\frac{1}{2}}}\right)\nabla_T^2 T \\ & + \left(\frac{d^2}{ds^2}\frac{1}{(\kappa^2+a^2)^{\frac{1}{2}}} + \frac{\kappa^2 - a^2}{(\kappa^2+a^2)^{\frac{1}{2}}}\right)\nabla_T T \\ & + \frac{d}{ds}\left(\frac{\kappa^2 - a^2}{(\kappa^2+a^2)^{\frac{1}{2}}}\right) T \\ & + \frac{1}{(\kappa^2+a^2)^{\frac{1}{2}}} R(\nabla_T T, T)T\,, \end{aligned} \qquad (10)$$

$T = e_0(s) = \gamma'$ being the unit tangent vector to γ, R denoting the Riemannian curvature tensor of M^m, while the boundary operator $\mathcal{B}$ is given by

$$\mathcal{B}[\gamma, W] = \langle \nabla_T W, \mathfrak{K}\rangle|_0^1 - \langle W, \mathfrak{J}\rangle|_0^1\,,$$

where

$$\begin{aligned} \mathfrak{K} &= \frac{1}{(\kappa^2+a^2)^{\frac{1}{2}}}\nabla_T T\,, \\ \mathfrak{J} &= \nabla_T \mathfrak{K} + \frac{\left(\kappa^2 - a^2\right)}{(\kappa^2+a^2)^{\frac{1}{2}}} T\,. \end{aligned} \qquad (11)$$

Then, from (9) one sees that a curve $\gamma \in \Omega$ satisfying suitable first order boundary conditions (for instance, closed curves) is a critical point of $\mathcal{F}_a$, if and only if,

$$\mathcal{E}(\gamma) = 0. \qquad (12)$$

and, consequently, geodesics are always extremals of $\mathcal{F}_a$.

Case $a = 0$ corresponds to the total curvature functional which has been already discussed in the introduction. So we assume $a \neq 0$.

Now we consider $\mathcal{F}_a$ acting on the subspace $\widetilde{\Omega} \subset \Omega$ which is formed by closed curves. One sees from (10) and (12) that *every closed geodesic is a critical point of* $\mathcal{F}_a$ in $\widetilde{\Omega}$. Hence we may assume in addition

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that $\gamma \in \widetilde{\Omega}$ is a non-geodesic closed curve, namely, that it is a curve of rank at least 1. As it has been noticed before, γ is an extremal, if and only if, $\mathcal{E}(\gamma) = 0$.

If $M^n(\rho)$ is a m-space real form with constant curvature ρ, then, it was shown in [4] that extremals fully lie in a totally geodesic submanifold of dimension three at most. So, we may assume $n \leq 3$ and we use the standard notation for the Frenet frame $\{e_0(s) \equiv T(s), e_1(s) \equiv N(s), e_2(s) \equiv B(s)\}$ (which will be referred to as *unit tangent*, *unit normal* and *binormal*, respectively) and Frenet curvatures $\{\kappa \equiv \kappa_1, \tau \equiv \kappa_2\}$ (which will be referred to as *curvature* and *torsion*, respectively). Using the well known expression for the curvature tensor $R(X,Y)Z = \rho\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\}$, the Frenet formulas (7), and the linear independence of the Frenet frame, the Euler-Lagrange equations (10) reduce, after some straightforward computations, to

$$\frac{d^2}{ds^2}\left(\frac{\kappa}{(\kappa^2+a^2)^{\frac{1}{2}}}\right) + \tag{13}$$

$$\frac{\kappa}{(\kappa^2+a^2)^{\frac{1}{2}}}\left(\kappa^2 - \tau^2 + \rho\right) - \kappa\left(\kappa^2+a^2\right)^{\frac{1}{2}} = 0\,,$$

$$\frac{d}{ds}\left(\frac{\kappa^2}{(\kappa^2+a^2)}\tau\right) = 0\,. \tag{14}$$

If $\rho > 0$, that is when we are considering curves in spheres with curvature ρ, $\mathbb{S}^n(\rho), n = 2, 3$, extremals of $\mathcal{F}_a$ are well known [4]

Proposition 1. *Let $\mathcal{F}_a : \widetilde{\Omega} \longrightarrow \mathbb{R}$ be the energy functional defined in (8) acting on $\widetilde{\Omega}$, the space of closed curves in $\mathbb{S}^3(\rho)$. Then, we have*

1. *If $a^2 > \rho$, then the only closed critical curves are the geodesics;*
2. *If $a^2 = \rho$, then the only closed critical curves are the circles;*
3. *If $0 < a^2 < \rho$, then*
   - (a) *in addition to geodesics, the set of closed critical points in $\mathbb{S}^2(\rho)$ forms a countable infinite family of curves with non-constant curvature.*
   - (b) *the set of closed critical helices fully immersed in $\mathbb{S}^3(\rho)$ forms a rational 1-parameter family.*

Thus, it remains to consider the case $\rho \leq 0$. If the geodesic curvature κ were constant, then (13) and (14) would give that τ is constant and $\varphi = \tau^2 + a^2$ which is impossible. So, there are no extremals with constant curvature. If κ is not constant, then one may check that the following equations are first integrals of (13) and (14)

$$\left(\frac{dk}{ds}\right)^2 = \left(\frac{a^2+\kappa^2}{a^2\kappa}\right)^2 \times \tag{15}$$

$$\left(\left(\kappa^2+a^2\right)\left(d\kappa^2 - e(\kappa^2+a^2)\right) - \kappa^2(\rho\kappa^2+a^4)\right)\,,$$

$$\tau = e\left(\frac{\kappa^2+a^2}{\kappa^2}\right)\,, \tag{16}$$

where d, e are constants of integration. Setting $u(s) = \kappa^2$, equation (15) can be written as

$$\left(\frac{d\,u}{ds}\right)^2 = \frac{4}{a^4}(u^2+a^2)^2\,Q(u)\,, \tag{17}$$

where $Q(x) = (d - e^2 - \rho)x^2 + (a^2 d - 2e^2a^2 - a^4)x - e^2a^4$. Periodic solutions of (13) and (14) would imply the existence of closed trajectories in the phase plane of the equation (17) for $u \geq 0$. An analysis of (17) shows that for this to happen we need $Q(u)$ to have two positive roots and positive value at any point between them. But the two roots of $Q(u)$ are $u = \frac{a^2(a^2+2e^2-d\pm\sqrt{(a^2-d)^2+4e^2(a^2-\rho)})}{2(d-e^2-\rho)}$, which can be check to be non-positive under the condition $\rho \leq 0$. The same reasoning applies equally well in surfaces $M^2(\rho)$. Hence we have

Proposition 2. *There are no closed extremals curves of $\mathcal{F}_a$ in $M^n(\rho)$, for constant curvature $\rho \leq 0$ and for any $a \in \mathbb{R} - \{0\}$.*

Extremals in $M^2(\rho)$ are totally determined by their curvature, κ, which, in our case, can be obtained explicitly. In fact, the Euler-Lagrange equation remains valid for critical curves with prescribed zero and first order boundary data. In particular, in surfaces with constant curvature ρ, $M^2(\rho)$, the Euler-Lagrange equations are obtained by taking $\tau = 0$ in (13) and (14). Hence, the geodesic curvature of extremals must satisfy (15) and (16) which, in this case, reduce to $\tau = e = 0$ and, after a little manipulation to

$$\left(\frac{dk}{ds}\right)^2 = \left(\frac{a^2+\kappa^2}{a^2}\right)^2\left(\kappa^2(d-\rho) + a^2(d-a^2)\right)\,. \tag{18}$$

If $\rho \leq 0$, this can be integrated to

$$\kappa(s) = \frac{a\sqrt{d-a^2}\,\tanh\left(\sqrt{a^2-\rho}\,s - c_1\right)}{\sqrt{a^2-\rho+(\rho-d)\,\tanh^2\left(\sqrt{a^2-\rho}\,s - c\right)}}\,. \tag{19}$$

$c \in \mathbb{R}$. Thus, a parametrization of the $\mathcal{F}_a$-extremals in $\mathbb{R}^2$, in terms of the arc-length parameter, is given by

$\left(\int\cos\int\kappa, \int\sin\int\kappa,\right)$, so it can be obtained after two quadratures using (19). This is a case relevant in image restoration as we mentioned at the beginning and a different parametrization was given in [11] assuming that the critical curves have no inflection points.

On the other hand, one may notice as a consequence of Proposition 1.(2), that every parallel of $\mathbb{S}^2(a^2)$ (as a surface of revolution in $\mathbb{R}^3$) is critical for $\mathcal{F}_a$. This fact can be generalized as follows.

For convenience, let us change notation a little bit for a while. Assume now that (M^3, g) is a Riemannian 3-manifold and let ξ be a *Killing vector field* on (M^3, g). Then ξ generates a one-parameter subgroup G_ξ of the group of isometries of M^3. An immersion $\varphi : S \to (M^3, g)$ of a surface S into M^3 is said to be a G_ξ*-equivariant* immersion, and $\varphi(S)$ a G_ξ*-invariant* surface of M^3, if there exists an action of G_ξ on S such that for any $x \in S$ and $h \in \mathrm{G}_\xi$ we have $\varphi(h\,x) = h\varphi(x)$. By pulling back the metric g via φ, a G_ξ-equivariant immersion $\varphi : S \to (M^3, g)$ induces a Riemannian metric on S, called the G_ξ-*invariant induced metric*, which is denoted by g_φ.

Endow S with the G_ξ-invariant induced metric g_φ, assume that $\varphi(S) \subset M^3_r$ (M^3_r denoting the regular part of M^3, namely, the subset consisting of points belonging to principal orbits), that M^3/G_ξ is connected, and equip M^3_r/G_ξ with the natural Riemannian metric making $\pi : M^3_r \to M^3_r/\mathrm{G}_\xi$ a Riemannian submersion, $\tilde{g}$ (for details, see, for instance, [25]).

Now, a local description of the G_ξ-invariant surfaces of can be given as follows. Let $\gamma : (\varepsilon_1, \varepsilon_2) \subset \mathbb{R} \to (M^3/\mathrm{G}_\xi, \tilde{g})$ be a curve parametrized by arc length and let $\tilde{\gamma} : (\varepsilon_1, \varepsilon_2) \subset \mathbb{R} \to M^3$ be a lift of γ, such that $d\pi(\tilde{\gamma}') = \gamma'$. If we denote by $\phi_v,\ v \in (-\epsilon, \epsilon)$, the local flow of the Killing vector field ξ, then the map

$$\begin{aligned} \psi : U \equiv (\varepsilon_1, \varepsilon_2) \times (-\epsilon, \epsilon) \to M^3\,, \\ \psi(s, v) = \phi_v(\tilde{\gamma}(s)), \end{aligned} \tag{20}$$

defines a parametrized G_ξ-invariant surface. Conversely, if $\varphi(S)$ is a G_ξ-invariant immersed surface in M^3, then $\gamma = \pi\varphi(S)$ defines a curve in $(M^3/\mathrm{G}_\xi, \tilde{g})$ that can be locally parametrized by arc length. The curve γ is generally called the *profile* curve of the invariant surface. From now on, we denote a G_ξ-invariant surface by S_γ, γ being its *profile* curve. We have

Proposition 3. *Let S_γ be G_ξ-invariant surface of M^3 all whose orbits are extremals curves of $\mathcal{F}_a$, for a given $a \in \mathbb{R}$, under suitable zero and first order boundary data. Then either every orbit is a geodesic of S_γ (and then S_γ is flat) or it has non-negative constant Gaussian curvature $K_{S_\gamma} = a^2$. Moreover, in the former case, the only closed extremals are closed geodesics (if any), while in the latter case the only closed extremals are closed curves with constant geodesic curvature (if any).*

Proof. Take $\tilde{\gamma}$ a horizontal lift of the profile curve of S_γ and consider the local parametrization of S_γ defined in (20). Let us define the following volume function on the orbits $\omega^2(s) := \|\xi(\tilde{\gamma}(s))\|^2_g$. Then, $\alpha_s(v) = \psi(s, \frac{v}{\omega(s)})$ is a unit speed parametrization of an orbit. Consider the two following orthonormal vector fields defined on any orbit $\alpha_s(v)$: $T_s(v) = \frac{d}{dv}\alpha_s(v)$, and $X_s(v)$, the ξ-invariant extension of $\frac{d}{ds}\tilde{\gamma}$. Then, after a suitable orientation if necessary, the geodesic curvature of $\alpha_s(v)$ in S_γ is given by $\kappa(v) = \langle\nabla_T T, X\rangle$ (for simplicity, we are using again $\langle,\rangle$ to denote the metric instead of g_φ). Now, ξ is a Killing field so

$$\begin{aligned} 2\omega(s)\frac{d\omega(s)}{ds} &= X\langle\xi,\xi\rangle = 2\langle\nabla_X\xi,\xi\rangle = -2\langle X, \nabla_\xi\xi\rangle \\ &= -2\omega^2(s)\langle X, \nabla_T T\rangle = -2\omega^2(s)\kappa_s(v)\,, \end{aligned}$$

therefore,

$$\kappa_s(v) = \frac{d\omega/ds}{\omega}(v)\,. \tag{21}$$

In other words, the geodesic curvature is constant on any orbit. Now, assume that $\forall s$ the orbit $\alpha_s(v)$ is an extremal of $\mathcal{F}_a$. Then, $\mathcal{E}(\gamma) = 0\ \forall s$ and applying (10) (11) we have

$$\begin{aligned} 0 = \frac{d^2}{dv^2}\left(\frac{\kappa_s}{(\kappa_s^2 + a^2)^{\frac{1}{2}}}\right) + \\ \frac{\kappa_s}{(\kappa_s^2 + a^2)^{\frac{1}{2}}}\left(\kappa_s^2 + K_{S_\gamma}\right) - \kappa_s\left(\kappa_s^2 + a^2\right)^{\frac{1}{2}}\,, \end{aligned} \tag{22}$$

and since the geodesic curvature is constant on orbits we get

$$\kappa_s(v)(K_{S_\gamma} - a^2)(s, v) = 0\,, \tag{23}$$

on U. Then, either the Gaussian curvature is constant on U, $K_{S_\gamma} = a^2$, or every orbit $\alpha_s(v)$ is a geodesic of S_γ. In the latter case, combining (21) and $K_{S_\gamma} = -\frac{d^2\omega(s)/ds^2}{\omega(s)}$, [25], we see that S_γ is flat. $\square$

Remark 4. *Observe that the orbits of S_γ are geodesics in S_γ, if and only if, $w(s)$ is constant along the profile curve. In this case, we say that S_γ is a ξ-cylinder shaped on the curve γ. Moreover, dim $S/\mathrm{G}_\xi = 1$ means that S_γ is a G_ξ-invariant surface of cohomogeneity one and they are known as generalized rotational surfaces [6]. Thus S/G_ξ is an open interval and if, in addition, the principal orbits are diffeomorphic to $\mathbb{S}^1$, they are known as generalized rotational* *surfaces of spherical type, [6]. The intrinsic geometry of a generalized rotational surface of spherical type is uniquely determined by the characteristic function $w(s)$ of orbit sizes.*

Corollary 5. *Generalized rotational surfaces of spherical type with constant Gaussian curvature a^2 are foliated by closed extremals of $\mathcal{F}_a$.*

In particular, the above corollary applies to rotation surfaces in real space forms with positive constant curvature. G_ξ-invariant surfaces with constant Gaussian curvature in homogeneous 3-manifolds have been studied in [14, 19, 24, 25].

4 Numerical Approach

We come back now to the problem described in section 2. As explained there, the problem of minimizing the functional (5) acting on the space of plane curves joining two given points of $\mathbb{R}^2$ with prescribed initial and final angles, can be linked to that of finding D-curves minimizing the length. Even better adapted to the curve completion demands is the variational problem associated to the functional $\mathcal{F}_a$, which has been discussed in the previous section. Now, we want to analyze this problem directly on the unit tangent bundle.

More precisely, we are led to the following variational problem. Denote by $\mathfrak{X}$ the space of curves

$$\beta\colon\ [a,b] \longrightarrow \mathbb{R}^2\times\mathbb{S}^1;\quad \beta\left(t\right)=\left(x\left(t\right),y\left(t\right),\theta\left(t\right)\right)$$

joining two given points (x_a, y_a, θ_a) and (x_b, y_b, θ_b) of $\mathbb{R}^2\times\mathbb{S}^1$, that is

$$\begin{aligned}\left(x\left(a\right),y\left(a\right),\theta\left(a\right)\right)&=\left(x_a,y_a,\theta_a\right)\ ,\\ \left(x\left(b\right),y\left(b\right),\theta\left(b\right)\right)&=\left(x_b,y_b,\theta_b\right)\ ,\end{aligned}\qquad (24)$$

and satisfying the following admissibility condition

$$y'(t)=x'(t)\tan\theta(t)\,,\quad t\in[a,b]\ .\qquad (25)$$

Now, consider the functional $\mathcal{L}$ defined on $\mathfrak{X}$ by

$$\mathcal{L}\left(\beta\right)=\int_a^b\sqrt{\left(x'\right)^2+\left(y'\right)^2+h^2\left(\theta'\right)^2}dt\,,\qquad (26)$$

where $'$ denotes derivative with respect to the parameter $t\in[a,b]$ and $h\in\mathbb{R}$ is proportionality constant introduced by accuracy of the physical model [11]. Then, one must find the minimizers of $\mathcal{L}:\mathfrak{X}\to\mathbb{R}$.

The corresponding Euler-Lagrange equations can be computed and solve in terms of elliptic functions [11]. However, specific determination of the solution curves requires solving a highly nonlinear system for which an explicit expression seems unlikely. Hence, a numerical approach is a reasonable strategy.

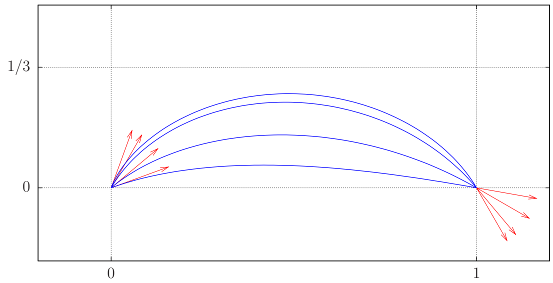

Figure 1: Minimizers of (26) obtained via $XEL-2.0$.

In [5] we have developed a gradient descent based method (which we call XEL-platform) to localize minima of an ample family of functionals defined on certain spaces of curves satisfying both affine and isoperimetric constrains. These spaces become Hilbert spaces under suitable Sobolev-type metrics and, also, the energy functionals are allowed to be weighted at the ends of the curves. Then, a numerical method to locate minimizers of this general class of variational problems is implemented in [5] (see also, www.ikergeometry.org).

In particular, for the functional $\mathcal{L}:\mathfrak{X}\to\mathbb{R}$ defined in (26) the XEL-platform can be applied. For this, we can take, without loss of generality, $a=0, b=1$ and the ends of the curves are chosen to be

$$p_0=(0,0,\theta_0)\quad,\quad p_1=(1,0,\theta_1)\ .$$

Figure 1 shows a family of minimizers for different choices of end angles θ_0,θ_1 detailed in Table 1. One may wish to compare extremals obtained by our with those got in [11], where the authors used a different numeric approach. For example, Table 1 also shows the length of the extremal curves obtained with both methods for identical initial data. Thus, although both numeric approaches are conceptually very different, it is remarkable that the results obtained by either method are very close concerning both shape and length of the extremals.

One advantage of our XEL-platform is that it is easily adaptable to a huge family of functionals satisfying the required conditions. For instance, it has been pointed out in [11] that investigation of extremals for the elastic energy $\hat{\mathcal{F}}:\mathfrak{X}\to\mathbb{R}$; $\hat{\mathcal{F}}(\beta)=\int_\gamma\kappa_\beta^2$, where κ_β denotes the geodesic curvatures of β, might be important for examining combinations of image plane properties. Having this in mind, we can proved that if

$$\alpha=\alpha\left(s\right)=\left(x(s),y(s)\right)$$

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is an arc-length parametrized plane curve, and β denotes its lift to $\mathfrak{X}$ (with slope $h \in \mathbb{R}$) then its elastic energy in $\mathbb{R}^2 \times \mathbb{S}^1$ is given by

$$\int_0^l \kappa_\beta^2 v_\beta ds = \int_0^l \frac{\kappa_\alpha^2(s)}{(1+h^2\kappa_\alpha^2)^{3/2}} ds + h^2 \int_0^l \frac{(\kappa_\alpha'(s))^2}{(1+h^2\kappa_\alpha^2)^{5/2}} ds,$$

where κ_β, κ_α denote the geodesic curvatures of β in $\mathbb{R}^2 \times \mathbb{S}^1$ and of α in $\mathbb{R}^2$, respectively.

	Length of the curve $\beta(s)$ in $\mathbb{R}^2 \times \mathbb{S}^1$	
θ_0, θ_1	[11]	XEL2-0
$\theta_0 = 20, \theta_1 = -10$	1.1687	1.1691
$\theta_0 = 40, \theta_1 = -30$	1.6287	1.6289
$\theta_0 = 60, \theta_1 = -50$	2.2435	2.2436
$\theta_0 = 70, \theta_1 = -60$	2.5758	2.5761

Table 1

Moreover, we have

$$\int_0^l \frac{\kappa_\alpha^2(s)\, ds}{(1+h^2\kappa_\alpha^2)^{3/2}} = \int_0^l \frac{\left(\frac{\theta'(t)}{v_\alpha(t)}\right)^2 v_\alpha dt}{\left(1+h^2\left(\frac{\theta'(t)}{v_\alpha(t)}\right)^2\right)^{3/2}} = \int_0^l \frac{(\theta'(t)\, v_\alpha(t))^2\, dt}{\left(v_\alpha^2(t) + h^2(\theta'(t))^2\right)^{3/2}},$$

$$\int_0^l \frac{(\kappa_\alpha'(s))^2\, ds}{(1+h^2\kappa_\alpha^2)^{5/2}} = \int_a^b \frac{\left(\frac{\theta''(t)}{v_\alpha^2(t)} - \frac{\theta'(t)v_\alpha'(t)}{v_\alpha^3(t)}\right)^2 v_\alpha}{\left(1+h^2\left(\frac{\theta'(t)}{v_\alpha(t)}\right)^2\right)^{5/2}} dt = \int_a^b \frac{(\theta''(t)\, v_\alpha - \theta' v_\alpha')^2\, dt}{\left(v_\alpha^2(t) + h^2(\theta'(t))^2\right)^{5/2}}.$$

which allows $\hat{\mathcal{F}}$ to be written in a way suitable for our numerical experiments with the XEL-platform.

In this respect, many different experiments can be performed with the elastic energy functional $\hat{\mathcal{F}}$ in $\mathbb{R}^2 \times \mathbb{S}^1$. For example, choosing again $p_0 = (0, 0, \theta_0)$, $p_1 = (1, 0, \theta_1)$ as end points for our curves, and several ending angles, θ_0, θ_1, one may wish to find extremals of the elastic energy for, say, variations with (or without) the same length, and/or with (or without) prescribed ending curvatures.

Thus, when there is no penalty on the length, Figure 2 shows extremals obtained under both conditions, prescribed or unprescribed ending curvatures,

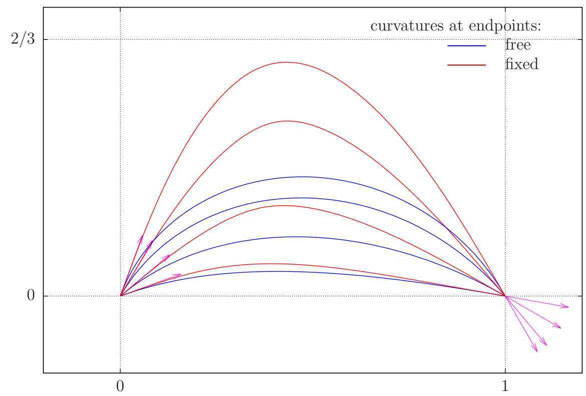

Figure 2: Minimizing the elastic energy.

and Table 2 shows initial data and the corresponding value of the elastic energy under both conditions (in this example, it is assumed also that $\theta(t) = \theta_0 + (\theta_1.\theta_0)\, t$).

	Elastic energy in $\mathbb{R}^2 \times \mathbb{S}^1$	
(θ_0, θ_1)	Prescr. curv.	Free
$(20, -10)$	2.84653	0.62874
$(40, -30)$	0.88453	0.46458
$(60, -50)$	0.56127	0.45739
$(70, -60)$	0.53310	0.46825

Table 2

5 Conclusions

Subriemannian geodesics in the unit tangent bundle $\mathbb{R}^2 \times \mathbb{S}^1$ play an important role in recent models for the primary visual cortex V1. It turns out that such geodesics have to be lifts to $\mathbb{R}^2 \times \mathbb{S}^1$ of curves in $\mathbb{R}^2$ which are critical for a total curvature type functional. The variational problem associated to this curvature energy is considered for curves in real spaces forms. In this case, we compute the first variation formula and first integrals of the corresponding Euler-Lagrange equations. Using them, we prove that there are no extremal curves if the sectional curvature of the ambient space in non-positive and we solve the Euler-Lagrange equations when the dimension of the ambient space is 2, what completely determines the extremals up to two quadratures. We also show that G_ξ-invariant surfaces of a 3-space (i.e., surfaces which are invariant under the 1-parameter group of isometries G_ξ associated to a Killing field ξ) whose orbits are all critical for this energy, must have non-negative constant Gaussian curvature.

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Finally, for practical purposes it is better to directly consider subriemannian geodesics in $\mathbb{R}^2 \times \mathbb{S}^1$ from a numerical point of view. We use a gradient descent based method implemented by our group somewhere else. For subriemannian geodesics our results are in close agreement with those obtained by other authors, however, our method is useful for a large family of functionals, as we show here by considering the elastic energy.

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