Relativistic Effects in a moving Point Charge

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Abstract: - Using the transformation law of the electromagnetic field between two inertial frames, the Lorentz force $\vec{F} = q\vec{V} \wedge \vec{B}$, that acts on a charged particle moving in a magnetic field, is obtained.

It is shown why this force, despite being of relativistic nature, manifests itself at any value of speed of the moving charge.

In addition, using the same transformation laws, the electromagnetic field generated by a charge that moves with constant speed, in the direction perpendicular to the speed, is calculated.

Key-Words: - physical science, magnetism, electricity, electrodynamics, electromagnetic induction, special relativity, fields transformations, Lorentz force.

1 Introduction

General Physics is usually taught at the first years of Engineering courses. Usually, the contents of this course are focused on General Mechanics and Electromagnetism, sometimes including Thermodynamics and Fluid Dynamics. It is rare to teach Special Relativity (SR), even if some basic aspects of this theory are recalled when introducing the Electromagnetic waves and fields.

The two basic postulates of SR are [1]:

P1. laws of physics are invariant in all inertial reference frames;

P2. the speed of light in a vacuum is constant and it is the same in all the reference frames.

In this paper, the authors, focusing on a didactic approach, will show how the Lorentz Force can be derived as a relativistic effect.

Finally is also calculated the transverse electromagnetic field generated by a moving point charge.

2 Lorentz Transformation

Let us consider two inertial reference frames S and S', that move with x and x' axes superimposed and y and y', z and z' parallel to each other. Let $\vec{V}$ be the constant speed, parallel to x and x', of frame S' with respect to frame S. Let us start timing from the instant in which the origins, O and O', of the two reference frames coincide.

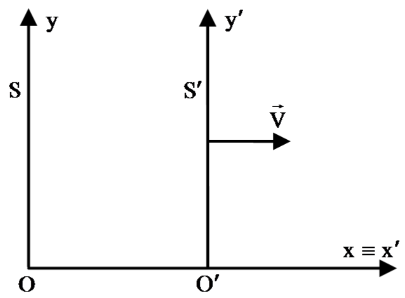

Fig. 1: S and S' are two inertial reference frames that move with a relative speed $\vec{V}$ (the axes z, z' are perpendicular to the plane and coming out of the paper).

Let (x, y, z, t) be an event in the reference frame S. The spatiotemporal coordinates of the same event seen from frame S' are:

$$\begin{cases} \dfrac{t'}{\gamma} = -\dfrac{V}{c^2}x + t & y' = y \\ \dfrac{x'}{\gamma} = x - Vt & z' = z \end{cases} \tag{1}$$

where: $\gamma = \dfrac{1}{\sqrt{1 - V^2/c^2}} > 1$, being *c* the speed of light in vacuum.

Relations (1) are known as Lorentz transformations and can be promply derived when one imposes the invariance, under these transformations, of the dalembertian operator:

$$\tilde{\nabla}^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

In our mono-dimensional case this invariance reads:

$$\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}$$

Also indicated in contracted form:

$$\tilde{\nabla}^2 = \tilde{\nabla}'^2.$$

The (1) have the following properties:

a) It is possible to write the inverse relations, related to the event (x', y', z', t') that occurs in the frame S', simply exchanging the superscript and posing $V \to -V$.

$\frac{1}{\gamma}\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -V \\ -V/c^2 & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}$	(2)
$\frac{1}{\gamma}\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 1 & V \\ V/c^2 & 1 \end{pmatrix}\begin{pmatrix} x' \\ t' \end{pmatrix}$	

b) Considering $\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} > 1$ we have that $V \geq c$ should give an imaginary γ, and so imaginary lengths, that does not have any physical meaning. This leads to the conclusion that the relative speed between two inertial reference frames cannot be equal to or greater than light speed.

c) The important relation holds:

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \;\to\; \text{Invariant}$$

That is easily explained using property a) and considering that the speed of light is the same in every reference frame.

d) For speeds, small with respect to light's, equations (1) give back the classical Galilean transformations:

$$\begin{cases} t' = t \quad (\text{absolute time}) & y' = y \\ x' = x - Vt & z' = z \end{cases}$$

3 Charge density and current density transformations

Let us consider a charge at rest in the inertial reference frame S', distributed according to the density:

$$\rho_0 = \frac{dq}{dx_0 dy_0 dz_0} \equiv \frac{dq}{d\eta_0} \;\to\; \boxed{dq = \rho_0\, d\eta_0}$$

where $d\eta_0$ is the elementary volume occupied by the charge at rest in the reference frame S'.

Let us study how the density transforms if the charge distribution is seen from the inertial reference frame S. In SR the charge is invariant, but lengths contract in the direction of the speed $\vec{V}$ of the frame S' relative to S, so:

$$\boxed{dx = \frac{dx_0}{\gamma}} \quad dy = dy_0 \quad dz = dz_0 \;\to\; \boxed{d\eta = \frac{d\eta_0}{\gamma}}$$

Then, the charge density, seen from S, is:

$$\rho = \frac{dq}{d\eta} = \gamma\frac{dq}{d\eta_0} = \gamma\rho_0. \qquad (3)$$

In frame S, the current density on the *x* axis is:

$$J_x = \rho w = \frac{dq}{d\eta} w = dq \frac{V}{d\eta_0/\gamma} = \gamma\rho_0 V \qquad (4)$$

where has been realized that charge, at rest in S', has a velocity w.r.t. S equal to w, which is the same as the relative velocity V.

Squaring (4):

$$J_x^2 = \gamma^2\rho_0^2 V^2$$

and subtracting from both sides ρ^2c^2, one has:

$$J_x^2 - \rho^2c^2 = \gamma^2\rho_0^2V^2 - \rho^2c^2$$

$$J_x^2 - \rho^2c^2 = \gamma^2\rho_0^2V^2 - \gamma^2\rho_0^2c^2 = \rho_0^2\gamma^2(V^2 - c^2)$$

But, being: $\gamma^2(V^2 - c^2) = c^2$ we have, at last:

$$J_x^2 - \rho^2c^2 = \rho_0^2 c^2$$

where the r.h.s. is an invariant quantity. So comparing this with

$$x^2 - c^2t^2 \;\to\; \text{Invariant}$$

we see that ρ transforms like t and J_x transforms like x. So, considering (2), we have:

$$\begin{cases} \dfrac{\rho'}{\gamma} = -\dfrac{V}{c^2} J_x + \rho & J'_y = J_y \\ \dfrac{J'_x}{\gamma} = J_x - V\rho & J'_z = J_z \end{cases} \qquad (5)$$

4 Electromagnetic potentials transformations

It is well known that the electromagnetic field can be deduced by electromagnetic potentials $\vec{A}$ and Φ, respectively magnetic vector potential and electric scalar potential, by means of the following relations:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \text{grad}\,\Phi \qquad \vec{B} = \text{rot}\,\vec{A} \qquad (6)$$

In the Lorentz gauge, the potentials equations are:

$$\begin{cases} \tilde{\nabla}^2 \vec{A} = -\mu_0 \vec{J} \\ \left(\tilde{\nabla}^2\right)\Phi = -\dfrac{\rho}{\varepsilon_0} \end{cases} \rightarrow \begin{cases} \tilde{\nabla}^2 \vec{A} = -\mu_0 \vec{J} \\ \left(\tilde{\nabla}^2\right)\dfrac{\Phi}{c^2} = -\mu_0 \rho \end{cases} \qquad (7)$$

having substituted: $1/\varepsilon_0 = \mu_0 c^2$.

Since the dalembertian $\tilde{\nabla}^2$ is invariant under a Lorentz transformation, and J_x, ρ transform in the same way of x, t we can deduce that A_x, Φ/c^2 have to transform as in (1):

$$\begin{cases} \dfrac{1}{\gamma}\dfrac{\Phi'}{c^2} = -\dfrac{V}{c^2} A_x + \dfrac{\Phi}{c^2} & A'_y = A_y \\ \dfrac{1}{\gamma} A'_z = A_x - V\dfrac{\Phi}{c^2} & A'_z = A_z \end{cases} \qquad (8)$$

5 Electromagnetic field transformations

Projecting (6) onto *x* axis, one has:

$$E_x = -\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x} \qquad B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \qquad (9)$$

Thus, the electromagnetic field components, taking into account (8) and

$\frac{1}{\gamma}\begin{pmatrix}\partial_{x'} \\ \partial_{t'}\end{pmatrix} = \begin{pmatrix}1 & +V/c^2 \\ +V & 1\end{pmatrix}\begin{pmatrix}\partial_x \\ \partial_t\end{pmatrix}$	(10)
$\frac{1}{\gamma}\begin{pmatrix}\partial_{x} \\ \partial_{t}\end{pmatrix} = \begin{pmatrix}1 & -V/c^2 \\ -V & 1\end{pmatrix}\begin{pmatrix}\partial_{x'} \\ \partial_{t'}\end{pmatrix}$	

transform as:

$$E'_x = -\frac{\partial A'_x}{\partial t'} - \frac{\partial \Phi'}{\partial x'} =$$

$$= \gamma^2\left(\frac{V^2}{c^2}\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x} + \frac{V^2}{c^2}\frac{\partial A_x}{\partial t}\right) =$$

$$= -\frac{1}{1 - V^2/c^2}\left[\left(1 - \frac{V^2}{c^2}\right)\frac{\partial \Phi}{\partial x} + \left(1 - \frac{V^2}{c^2}\right)\frac{\partial A_x}{\partial t}\right] =$$

$$= -\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x} = E_x$$

$$E'_y = -\frac{\partial A'_y}{\partial t'} - \frac{\partial \Phi'}{\partial y'} =$$

$$= \gamma\left[\left(-\frac{\partial A_y}{\partial t} - \frac{\partial \Phi}{\partial y}\right) - V\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right] = \gamma\left(E_y - VB_z\right)$$

$$E'_z = -\frac{\partial A'_z}{\partial t'} - \frac{\partial \Phi'}{\partial z'} =$$

$$= \gamma\left[\left(-\frac{\partial A_z}{\partial t} - \frac{\partial \Phi}{\partial z}\right) + V\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right] = \gamma\left(E_z + VB_y\right)$$

$$B'_x = \frac{\partial A'_z}{\partial y'} - \frac{\partial A'_y}{\partial z'} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = B_x$$

$$B'_y = \frac{\partial A'_x}{\partial z'} - \frac{\partial A'_z}{\partial x'} =$$

$$= \gamma\left[\frac{\partial A_x}{\partial z} - \frac{V}{c^2}\frac{\partial \Phi}{\partial z} - \frac{\partial A_z}{\partial x} - \frac{V}{c^2}\frac{\partial A_z}{\partial t}\right] =$$

$$= \gamma\left[\frac{V}{c^2}\left(-\frac{\partial A_z}{\partial t} - \frac{\partial \Phi}{\partial z}\right) + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right] =$$

$$= \gamma\left(\frac{V}{c^2}E_z + B_y\right)$$

$$B'_z = \frac{\partial A'_y}{\partial x'} - \frac{\partial A'_x}{\partial y'} =$$

$$= \gamma\left[\frac{\partial A_y}{\partial x} + \frac{V}{c^2}\frac{\partial A_y}{\partial t} - \frac{\partial A_x}{\partial y} + \frac{V}{c^2}\frac{\partial \Phi}{\partial y}\right] =$$

$$= \gamma\left[-\frac{V}{c^2}\left(-\frac{\partial A_y}{\partial t} - \frac{\partial \Phi}{\partial y}\right) + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right] =$$

$$= \gamma\left(-\frac{V}{c^2}E_y + B_z\right)$$

Summarizing:

$$\begin{cases} E'_x = E_x \\ \dfrac{1}{\gamma}E'_y = E_y - VB_z \\ \dfrac{1}{\gamma}E'_z = E_z + VB_y \end{cases} \qquad \begin{cases} B'_x = B_x \\ \dfrac{1}{\gamma}B'_y = B_y + \dfrac{V}{c^2}E_z \\ \dfrac{1}{\gamma}B'_z = B_z - \dfrac{V}{c^2}E_y \end{cases} \qquad (11)$$

whose inverse transformations can be obtained exchanging the superscripts and substituting $V \to -V$.

6 Lorentz Force as a relativistic effect

Let us consider a uniform magnetic field $\vec{B}$, with a charged particle q, moving with constant speed $\vec{V}$, perpendicular to $\vec{B}$.

Let S be the inertial frame with *x* axis in the same direction of $\vec{V}$ and *z* axis parallel to $\vec{B}$. In this frame, the vectors $\vec{B}$ and $\vec{V}$ have components:

$$\vec{B} = B\hat{k} \qquad \vec{V} = V\hat{i}$$

In addition, let S' be the inertial frame, moving with the charged particle and translating with respect to S with speed $\vec{V}$, having its x' axis overlapped to *x* axis.

Moving from an inertial frame to the other, the fields transform as (11), and being, now, in S only a magnetic field: $\vec{E} = \vec{0}$, $\vec{B} = B\hat{k}$, we find out from (11) that:

$$\begin{cases} E'_x = 0 \\ E'_y = -\gamma VB \\ E'_z = 0 \end{cases} \quad \begin{cases} B'_x = 0 \\ B'_y = 0 \\ B'_z = \gamma B \end{cases} \quad \to \quad \begin{cases} \vec{E}' = (-\gamma VB)\hat{j} \\ \\ \vec{B}' = (\gamma B)\hat{k} \end{cases}$$

So in the frame S', two fields are present: a magnetic field $\vec{B}'$ (uniform, in the z' axis direction and increased by a factor of γ) and an electric field $\vec{E}'$, (uniform, in the opposite direction of y' axis), also increased by a factor of γ and created just as a relativistic effect.

The charged particle q, at rest in the frame S', feels the presence of the electric field $\vec{E}'$ as a force:

$$\vec{F}' = q\vec{E}' = (-\gamma qVB)\,\hat{j}$$

Now, the point is the evaluation of this force in the frame S. More precisely, we have to evaluate the transformation of the force component perpendicular to the speed of the frame S' with respect to S.

For this reason, let us suppose that the particle in the frame S has a given impulse $dp_\perp$ perpendicular to the speed $\vec{V}$. This impulse does not change if evaluated from the frame S':

$$dp_\perp = dp'_\perp$$

On the contrary, the time dt of the frame S is different when evaluated by S', because of the effect of the time dilation:

$$dt = \gamma d\tau$$

Then:

$$F_\perp = \frac{dp_\perp}{dt} = \frac{1}{\gamma}\frac{dp'_\perp}{d\tau} = \frac{1}{\gamma}F'_\perp$$

In the frame S, the force acting on the charged particle is then:

$$\vec{F} = \frac{1}{\gamma}(-\gamma qVB)\hat{j} \quad \to \quad \vec{F} = (-qVB)\hat{j}$$

which is independent of γ !.

In addition, because:

$$qV\hat{i} \wedge B\hat{k} = qVB\ (\hat{i} \wedge \hat{k}) = qVB\,(-\hat{j})$$

It is evident the previous relation can be written as:

$$\vec{F} = q\vec{V} \wedge \vec{B}$$

that is exactly the Lorentz force.

Finally, let us observe that this force exists also if the speed of the charged particle is small with respect to the one of light. In this case, the magnetic field seen from the frame S' is practically the same of that seen in the frame S.

7 Field generated by a point charged particle with constant speed, in the direction perpendicular to the speed

With respect to the inertial frame S, let us consider a point charge q that moves with constant speed $\vec{V}$, parallel to the *x* axis. Let also S' be the inertial frame moving with the charge, placed in its origin.

In the frame S', in a point P of the y' axis, at a distance d' from the charge q (i.e. in the radial direction with respect to the charge and perpendicular to the speed $\vec{V}$), there is an electric field but not a magnetic field.

$$\vec{E}' = \frac{1}{4\pi\varepsilon_0}\frac{q}{d'^2}\hat{j} \qquad \vec{B}' = \vec{0}$$

In the frame S, considering that $d' = d$ (there is no contraction of length in the direction perpendicular to the speed $\vec{V}$) and that the electric charge is invariant, according to (11) we have:

$$\vec{E} = \gamma\frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}\hat{j} \qquad \vec{B} = \gamma\frac{1}{4\pi\varepsilon_0}\frac{Vq}{c^2d^2}\hat{k}$$

Thus, the charge in motion, beside the electric field (greater than the one generated in the frame in which it is at rest), generates also a magnetic field.

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Let us observe that $\vec{E}$ and $\vec{B}$ are orthogonal and the following relation holds between their moduli:

$$B = \frac{V}{c^2} E$$

In non relativistic conditions, i.e. when $V \ll c$, there is no differences between what is observed in the frame S and in the frame S', according to the expectations.

8 Conclusions

Using the transformation law of the electromagnetic field between two inertial frames S and S', the Lorentz force $\vec{F} = q\vec{V} \wedge \vec{B}$, that acts on a charged particle that moves in a magnetic field is obtained.

It is shown why this force, despite being of relativistic nature, manifests itself at any value of speed of the moving charge.

In addition, using the same transformation laws, the electromagnetic field generated by a charge that moves with constant speed, in the direction perpendicular to the speed, is calculated.

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Appendix

To facilitate controls of relations (10), (11), etc. and to let the readers acquire a unifying point of view we write here the matrices for transforming variables and their derivatives. At this aim, we start from relations (2):

$\frac{1}{\gamma}\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -V \\ -V/c^2 & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}$	(2)
$\frac{1}{\gamma}\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 1 & V \\ V/c^2 & 1 \end{pmatrix}\begin{pmatrix} x' \\ t' \end{pmatrix}$	

and, using the partial differentiation rules, one can write, for instance:

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x'} =$$

$$= \gamma\left(\frac{\partial}{\partial x} + \frac{V}{c^2}\frac{\partial}{\partial t}\right)$$

Similarly one can get and collect all main relations in the following matrix form:

$\frac{1}{\gamma}\begin{pmatrix} \partial_{x'} \\ \partial_{t'} \end{pmatrix} = \begin{pmatrix} 1 & +V/c^2 \\ +V & 1 \end{pmatrix}\begin{pmatrix} \partial_x \\ \partial_t \end{pmatrix}$	(10)
$\frac{1}{\gamma}\begin{pmatrix} \partial_x \\ \partial_t \end{pmatrix} = \begin{pmatrix} 1 & -V/c^2 \\ -V & 1 \end{pmatrix}\begin{pmatrix} \partial_{x'} \\ \partial_{t'} \end{pmatrix}$	

that, in intrinsic form (vectors-matrices) can be rewritten as:

$\frac{1}{\gamma}\hat{r}' = \overleftrightarrow{A}_{+V}\ \hat{r}$	$\frac{1}{\gamma}\partial_{\hat{r}'} = \overleftrightarrow{A}^{T}_{-V}\ \partial_{\hat{r}}$
$\frac{1}{\gamma}\hat{r} = \overleftrightarrow{A}_{-V}\ \hat{r}'$	$\frac{1}{\gamma}\partial_{\hat{r}} = \overleftrightarrow{A}^{T}_{+V}\ \partial_{\hat{r}'}$

This latter form has the advantage of showing - comparing the columns - that:

- To go from the coordinates of a reference frame to the ones of the other, it is enough to exchange the superscript and reverse the speed;

and, comparing the rows, that:

- To go from the coordinates of a frame to the partial derivatives related to the same reference frame, it is enough to transpose the matrix and reverse the speed.