On Approximations by Integro-Differential Splines of Two Variables

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Abstract: Here we construct the basic polynomial and exponential splines which can be used for plotting surfaces. A surface can be created if the values of the function in nodes and the values of the integrals are known.

Key–Words: Polynomial splines, Exponential splines, Integro-Differential Splines, Interpolation.

1 Introduction

Nowadays there are many different splines for solving different problems [1–8]. Polynomial integro-differential splines were first used by Kireev V.I. [9]. Integro-differential nonpolynomial approximations of one variable were suggested in [10, 11].

Local splines are useful for plotting surfaces. Here a surface is created if the values of the function in nodes and the values of the integrals are known.

2 Construction of the approximation

Let n, m be integer numbers, such that $n \geq 2$, $m \geq 1$, Let a, b, c, d be real numbers. Let us consider a rectangular domain $\bar{\Omega} = \Omega \bigcup \Gamma$ where

$$\Omega = \{(x, y) | a < x < b, c < y < d\}$$

and Γ is the boundary of Ω. We introduce $\Delta_x : a = x_0 < x_1 < \ldots < x_{n+1} = b$, $\Delta_y : c = y_0 < y_1 < \ldots < y_{m+1} = d$. We introduce a mesh of lines on $\bar{\Omega}$ which divides the domain $\bar{\Omega}$ into the rectangles $\bar{\Omega}_{j,k} = \Omega_{j,k} \bigcup \Gamma_{j,k}$,

$$\Omega_{j,k} = \{(x, y) | x \in (x_j, x_{j+1}), y \in (y_k, y_{k+1})\},$$

$\Gamma_{j,k}$ is the boundary of $\Omega_{j,k}$, $j = 0, \ldots, n$, $k = 0, \ldots, m$, $h_j = x_{j+1} - x_j$, $h_k = y_{k+1} - y_k$.

We denote $u_{j,k} = u(x_j, y_k)$. We associate the mesh $\Delta_x \times \Delta_y$ with the data: $(x_j, y_k, u_{j,k})$, $j = 0, 1, \ldots, n$, $k = 0, 1, \ldots, m$. It is supposed we know

$$I^{<0>}_{j,k} = \iint\limits_{\bar{\Omega}_{j,k}} u(x, y) dx dy,$$

$$I^{<-1>}_{j,k} = \int_{x_{j-1}}^{x_j} \int_{y_k}^{y_{k+1}} u(x, y) dx dy.$$

We construct an approximation $\tilde{u}(x, y)$ of $u(x, y)$ in $\bar{\Omega}_{j,k}$ in the form:

$$\tilde{u}(x, y) = u_{j,k} W_1(x, y) + u_{j+1,k} W_2(x, y) +$$
$$+ u_{j,k+1} W_3(x, y) + u_{j+1,k+1} W_4(x, y) +$$
$$+ I^{<0>}_{j,k} W_5(x, y) + I^{<-1>}_{j,k} W_6(x, y). \quad (1)$$

where basic splines $W_i(x, y)$ we obtain from the relations:

$$\tilde{u}(x, y) = u(x, y) \text{ for } u(x, y) = 1, x, y, xy, x^2, y^2. \quad (2)$$

Using the Tailor's formula

$$u(x, y) = u(x_j, y_k) + (x - x_j) u'_x(x_j, y_k) +$$
$$+ (y - y_k) u'_y(x_j, y_k) + \frac{1}{2!}\{(x - x_j)^2 u''_{xx}(x_j, y_k) +$$
$$2(x - x_j)(y - y_k) u''_{xy}(x_j, y_k) + (y - y_k)^2 u''_{yy}(x_j, y_k)\} + r,$$
$$r = \frac{1}{3!}\{(x - x_j)^3 u'''_{xxx}(s) + 3(x - x_j)^2 (y - y_k) u'''_{xxy}(s) +$$
$$+ 3(x - x_j)(y - y_k)^2 u'''_{xyy}(s) + (y - y_k)^3 u'''_{yyy}(s)\},$$

$s = (x_j + \tau(x_{j+1} - x_j), y_k + \tau(y_{k+1} - y_k))$, $\tau \in [0, 1]$, we can obtain $W_i(x, y)$ from the system of equations:

$$W_1(x, y) + W_2(x, y) + W_3(x, y) +$$
$$+ W_4(x, y) + I^{<0>}_{j,k} W_5(x, y) + I^{<-1>}_{j,k} W_6(x, y) = 1,$$
$$h_j W_2(x, y) + h_j W_4(x, y) + \iint\limits_{\bar{\Omega}_{j,k}} (x - x_j) dx dy W_5(x, y) +$$
$$+ \int\limits_{x_{j-1}}^{x_j} \int\limits_{y_k}^{y_{k+1}} (x - x_j) dx dy W_6(x, y) = x - x_j,$$

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$$h_kW_3(x,y)+h_kW_4(x,y)+\iint\limits_{\bar{\Omega}_{j,k}}(y-y_k)dxdy+$$

$$+W_5(x,y)+\int\limits_{x_{j-1}}^{x_j}\int\limits_{y_k}^{y_{k+1}}(y-y_k)dxdyW_6(x,y)=y-y_k,$$

$$h_j^2W_2(x,y)+h_j^2W_4(x,y)+\iint\limits_{\bar{\Omega}_{j,k}}(x-x_j)^2dxdyW_5(x,y)+$$

$$+\int\limits_{x_{j-1}}^{x_j}\int\limits_{y_k}^{y_{k+1}}(x-x_j)^2dxdy\,W_6(x,y)=(x-x_j)^2,$$

$$h_jh_kW_3(x,y)+h_jh_kW_4(x,y)+$$

$$+\iint\limits_{\bar{\Omega}_{j,k}}(x-x_j)(y-y_k)dxdy\,W_5(x,y)+$$

$$+\int\limits_{x_{j-1}}^{x_j}\int\limits_{y_k}^{y_{k+1}}(x-x_j)(y-y_k)dxdyW_6(x,y)=(x-x_j)(y-y_k),$$

$$h_k^2W_3(x,y)+h_k^2W_4(x,y)+\iint\limits_{\bar{\Omega}_{j,k}}(y-y_k)^2dxdy\,W_5(x,y)+$$

$$+\int\limits_{x_{j-1}}^{x_j}\int\limits_{y_k}^{y_{k+1}}(y-y_k)^2dxdy\,W_6(x,y)=(y-y_k)^2.$$

The value of determinant of the system is the next: $D=-(1/6)h_j^6h_k^6$. We obtain:

$W_1(x,y)=-1/(2h_k^2h_j^2)(-4h_j^2y^2-4h_j^2h_k^2k^2+6h_kyh_j^2-6h_k^2kh_j^2+xh_jh_k^2-jh_j^2h_k^2+x^2h_k^2+j^2h_j^2h_k^2+8h_j^2h_kyk-2h_jh_kxy+2h_jh_k^2xk+2h_j^2h_kjy-2h_j^2h_k^2jk-2xjh_jh_k^2-2h_j^2h_k^2)$,

$W_2(x,y)=1/(2h_k^2h_j^2)(xh_jh_k^2-jh_j^2h_k^2+x^2h_k^2-2xjh_jh_k^2+j^2h_j^2h_k^2-2h_jh_kxy+2h_jh_k^2xk+2h_j^2h_kjy-2h_j^2h_k^2jk+2h_j^2y^2-4h_j^2h_kyk+2h_j^2h_k^2k^2-2h_kyh_j^2+2h_k^2kh_j^2)$,

$W_3(x,y)=-1/(2h_k^2h_j^2)(x^2h_k^2-2xjh_jh_k^2+j^2h_j^2h_k^2-4h_j^2y^2+8h_j^2h_kyk-4h_j^2h_k^2k^2-xh_jh_k^2+jh_j^2h_k^2+2h_kyh_j^2-2h_k^2kh_j^2+2h_jh_kxy-2h_jh_k^2xk-2h_j^2h_kjy+2h_j^2h_k^2jk)$,

$W_4(x,y)=1/(2h_k^2h_j^2)(x^2h_k^2-2xjh_jh_k^2+j^2h_j^2h_k^2+2h_j^2y^2-4h_j^2h_kyk+2h_j^2h_k^2k^2-2h_kyh_j^2+2h_k^2kh_j^2-xh_jh_k^2+jh_j^2h_k^2+2h_jh_kxy-2h_jh_k^2xk-2h_j^2h_kjy+2h_j^2h_k^2jk)$,

$W_5(x,y)=-1/(h_j^3h_k^3)(5h_j^2y^2+5h_j^2h_k^2k^2-10h_j^2h_kyk-2xjh_jh_k^2-5h_kyh_j^2+5h_k^2kh_j^2+x^2h_k^2+j^2h_j^2h_k^2-xh_jh_k^2+jh_j^2h_k^2)$,

$W_6(x,y)=1/(h_j^3h_k^3)(-h_j^2y^2-h_j^2h_k^2k^2+2h_j^2h_kyk-2xjh_jh_k^2+h_kyh_j^2-h_k^2kh_j^2+x^2h_k^2+j^2h_j^2h_k^2-xh_jh_k^2+jh_j^2h_k^2)$.

If $h_k=h_j=h$, $x=x_j+th$, $y=x_k+t_1h$ then we have $W_1(x_j+th,y_k+t_1h)=-(1/2)t^2+2t_1^2-3t_1-(1/2)t+tt_1+1$,

$W_2(x_j+th,y_k+t_1h)=-tt_1+t_1^2-t_1+(1/2)t^2+(1/2)t$,

$W_3(x_j+th,y_k+t_1h)=-tt_1+2t_1^2-t_1-(1/2)t^2+(1/2)t$,

$W_4(x_j+th,y_k+t_1h)=tt_1+t_1^2-t_1+(1/2)t^2-(1/2)t$,

$W_5(x_j+th,y_k+t_1h)=-(1/h^2)(5t_1^2-5t_1+t^2-t)$,

$W_6(x_j+th,y_k+t_1h)=(1/h^2)(-t_1^2+t_1+t^2-t)$.

Figure 1 shows the basic functions $W_1(x,y)$, $W_2(x,y)$, $h=1$. Figure 2 shows the basic functions $W_3(x,y)$, $W_4(x,y)$, $h=1$. Figure 3 shows the basic functions $W_5(x,y)$, $W_6(x,y)$, $h=1$.

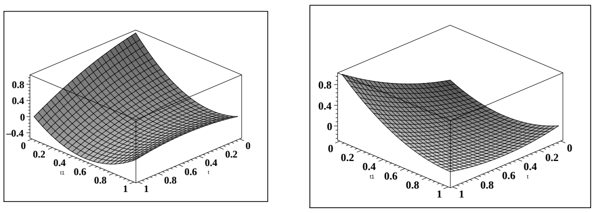

Figure 1: Plots of $W_1(x,y)$ (left), $W_2(x,y)$ (right)

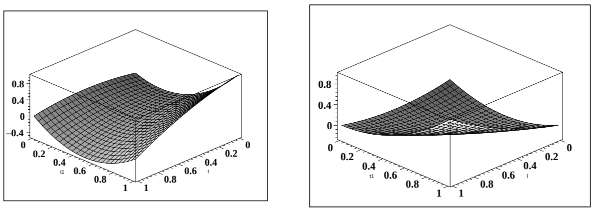

Figure 2: Plots of $W_3(x,y)$ (left), $W_4(x,y)$ (right)

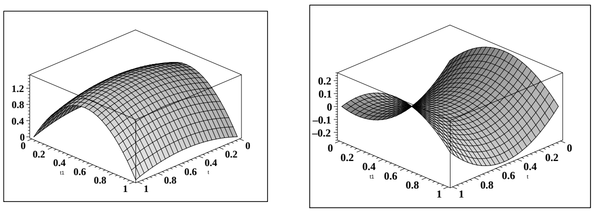

Figure 3: Plots of $W_5(x,y)$ (left), $W_6(x,y)$ (right)

Example 1. Let us take $\bar{\Omega}=[-0.5,0.4]\times[-0.5,0.4]$, $h=0.1$.

Table 1 shows the error of approximation: $\max_{j,k}|\widetilde{u}(s_{j,k})-u(s_{j,k})|$, $s_{j,k}=(x_j+0.05,y_k+0.05)$. Calculations were made in Maple with Digits=10.

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Table 1.

N	$u(x,y)$	$\max\limits_{j,k} \lvert\tilde{u}(s_{j,k}) - u(s_{j,k})\rvert$
1	x^2y^2	$0.208334e-5$
2	$(\sin(x)\cos(y))^2$	$0.247076e-5$
3	x^4y^4	$0.319319e-5$
4	$\sin(3x+3y)$	$0.502999e-4$
5	$\sin(5x+5y)$	$0.385416e-3$

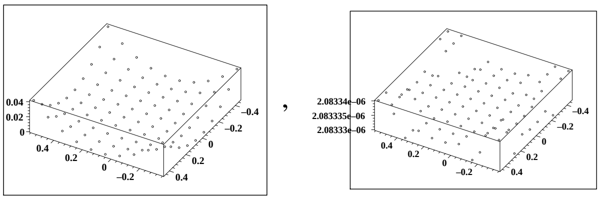

Figure 4: Plots of $\widetilde{u}(s_{j,k})$ (left) and $\widetilde{u}(s_{j,k}) - u(s_{j,k})$ (right), $u(x,y) = x^2y^2$

Figure 4 shows $\widetilde{u}(s_{j,k})$ and $\widetilde{u}(s_{j,k}) - u(s_{j,k})$ in Ω if $u(x,y) = x^2y^2$. Figure 5 shows $\widetilde{u}(s_{j,k})$ and $\widetilde{u}(s_{j,k}) - u(s_{j,k})$ in Ω if $u(x,y) = \sin(5x+5y)$.

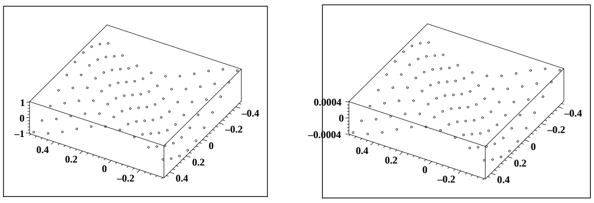

Figure 5: Plots of $\widetilde{u}(s_{j,k})$ (left) and $\widetilde{u}(s_{j,k}) - u(s_{j,k})$ (right), $u(x,y) = \sin(5x+5y)$

Figure 6 shows $\widetilde{u}(s_{j,k}) - u(s_{j,k})$ in Ω if $u(x,y) = (\sin(x)\cos(y))^2$, $u(x,y) = \sin(3x+3y)$.

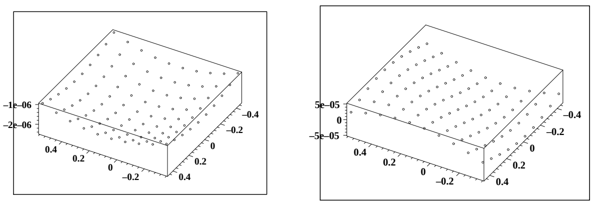

Figure 6: Plots of $\widetilde{u} - u$, $u(x,y) = (\sin(x)\cos(y))^2$ (left), $u(x,y) = \sin(3x+3y)$(right)

3 Second order approximation

Now it is supposed we know

$$I^{<0>}_{j,k} = \iint\limits_{\bar{\Omega}_{j,k}} u(x,y)dxdy.$$

1) We construct an approximation $\tilde{u}(x,y)$ of $u(x,y)$ in Ω_{jk} in the form:

$$\tilde{u}(x,y) = u(x_j,y_k)W_{1,j,k}(x,y) +$$

$$+u(x_{j+1},y_k)W_{2,j,k}(x,y) + I^{<0>}_{j,k}W^{<0>}_{j,k}(x,y),$$

where basic splines $W_{1,j,k}(x,y)$, $W_{2,j,k}(x,y)$, $W^{<0>}_{j,k}(x,y)$ we obtain from the relations:

$$\tilde{u}(x,y) = u(x,y) \text{ for } u(x,y) = 1, x, y.$$

Using the Tailor's formula

$$u(x,y) = u(x_j,y_k) + (x-x_j)u'_x(x_j,y_k) +$$

$$+(y-y_k)u'_y(x_j,y_k) + r,$$

where

$r = \frac{1}{2!}\{(x-x_j)^2u''_{xx}(s) + 2(x-x_j)(y-y_k)u''_{xy}(s) + (y-y_k)^2u''_{yy}(s)\}$,

$s = (x_j + \tau(x_{j+1}-x_j), y_k + \tau(y_{k+1}-y_k))$, $\tau \in [0,1]$,

we obtain $W_{1,j,k}(x,y), W_{2,j,k}(x,y), W^{<0>}_{j,k}(x,y)$ from the system of equations:

$$W_{1,j,k}(x,y) + W_{2,j,k}(x,y) + I^{<0>}_{j,k}W^{<0>}_{j,k}(x,y) = 1,$$

$$h_jW_{2,j,k}(x,y) + \iint\limits_{\bar{\Omega}_{j,k}} (x-x_j)dxdyW^{<0>}_{j,k}(x,y) = x - x_j,$$

$$\iint\limits_{\bar{\Omega}_{j,k}} (y-y_k)dxdyW^{<0>}_{j,k}(x,y) = y - y_k.$$

The value of the determinant of the system is the next: $D = h_j^2h_k^2/2$.

We find:

$W_{1,j,k}(x,y) = (h_jh_k - h_kx + h_jh_kj - h_jy + h_kh_jk)/(h_jh_k)$,

$W_{2,j,k}(x,y) = -(-h_kx + h_jh_kj + h_jy - h_kh_jk)/(h_jh_k)$,

$W^{<0>}_{j,k}(x,y) = -2(-y + kh_k)/(h_k^2h_j)$,

If $h_j = h_k = h$, and we put $x = x_j + th, y = y_k + t_1h$ then we have $W_{1,j,k}(x_j + th, y_k + t_1h) = 1 - t - t_1,\ t, t_1 \in [0,1]$,

$W_{2,j,k}(x_j + th, y_k + t_1h) = t - t_1$.

$W^{<0>}_{j,k}(x_j + th, y_k + t_1h) = 2t_1/h^2$.

2) We construct an approximation $\tilde{\tilde{u}}(x,y)$ of $u(x,y)$ in Ω_{jk} in the form:

$$\tilde{\tilde{u}}(x,y) = u(x_j,y_k)W_{1,j,k}(x,y) +$$

$$+u(x_{j+1},y_k)W_{2,j,k}(x,y) + I^{<0>}_{j,k}W^{<0>}_{j,k}(x,y),$$

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where basic splines $W_{1,j,k}(x,y)$, $W_{2,j,k}(x,y)$, $W_{j,k}^{<0>}(x,y)$ we obtain from the relations:

$$\tilde{u}(x,y) = u(x,y) \text{ for } u(x,y) = 1, e^x, e^y.$$

Example 2. Let us take $\bar{\Omega} = [-0.5, 0.4] \times [-0.5, 0.4]$, $h = 0.1$. Figure 7 shows $\widetilde{u}(s_{j,k}) - u(s_{j,k})$ in Ω if $u(x,y) = \sin(5x + 5y)$, $u(x,y) = x^4y^4$.

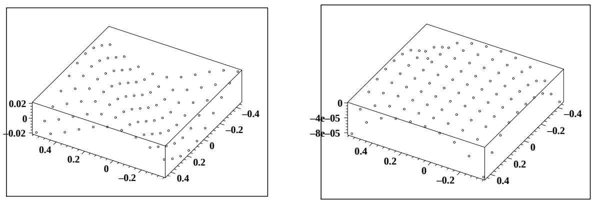

Figure 7: Plots of $\widetilde{u} - u$, $u(x,y) = \sin(5x+5y)$ (left), $u(x,y) = x^4y^4$ (right)

Table 2 shows the errors of approximations $\max_{j,k} |\widetilde{u}(s_{j,k}) - u(s_{j,k})|$ and $\max_{j,k} |\widetilde{\widetilde{u}}(s_{j,k}) - u(s_{j,k})|$ of the function $u(x,y)$, $s_{j,k} = (x_j + 0.05, y_k + 0.05)$, $\bar{\Omega} = [-0.5, 0.4] \times [-0.5, 0.4]$, $h = 0.1$.

Table 2.

N	$u(x,y)$	$\max \lvert \widetilde{u} - u \rvert$	$\max \lvert \widetilde{\widetilde{u}} - u \rvert$
1	x^2y^2	$0.33819e{-3}$	$0.24576e{-2}$
2	$\sin^2(x)\cos^2(y)$	$0.82393e{-3}$	$0.91366e{-3}$
3	x^4y^4	$0.16815e{-2}$	$0.82013e{-1}$
4	$\sin(3x + 3y)$	$0.74588e{-2}$	$0.81701e{-2}$
5	$\sin(5x + 5y)$	$0.20609e{-1}$	$0.21895e{-1}$
6	e^xe^y	$0.20504e{-2}$	$0.49988e{-4}$
7	$e^x + e^y$	$0.13071e{-2}$	$0.$

4 Conclusion

If the values of the integrals are unknown we can use cubature formulas. The order of the approximation can be obtained using the Tailor's formula.

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