On Integro-Differential Splines and Solution of Cauchy Problem

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Abstract: Here we construct implicit numerical methods for solving Cauchy problem with using polynomial and nonpolynomial integro-differential splines. Integro-differential polynomial splines were suggested in works of Kireev. In this case splines contain the values of integrals over net intervals. Integro-differential nonpolynomial splines were suggested by the author.

Key–Words: Splines, Interpolation, Integro-differential splines, Cauchy problem

1 Introduction

A large part of scientific computing is concerned with the solution of differential equations. Polynomial interpolation is quite useful for construction numerical methods for both ordinary and partial differential equations, especially boundary-value problems [1–3, 11].

Minimal interpolation splines were investigated in details in [4]. The distinctive feature of this splines is the existence of interpolation basis. The support of the basis spline contains one or several net intervals. This splines convenient for approximation functions and its derivatives with given error of approximation. Minimal interpolation splines suitable for solving interpolation problems of Lagrange, Hermit, Hermit-Birkhoff. The solution is constructed as the sum of products of the values of the function in the points of interpolation and the values of basic functions (and may be the values of their derivatives) on every net interval separately.

Integro-differential polynomial splines were suggested in works of Kireev. In this case splines contain the values of integrals over net intervals.

Here we construct numerical methods for solving Cauchy problem with using polynomial and polynomial integro-differential splines. Numerical methods for solving Cauchy problem with using minimal splines without values of integrals were suggested by the author in [5]. Some results were presented in [6, 8–10].

2 On non-polynomial integro-differential spline construction

Let $\alpha, m, m_\alpha, l_\alpha, s_\alpha, n, p_2, q$ — be integer nonnegative numbers, $l_\alpha \geq 1, s_\alpha \geq 1$, $m_\alpha = s_\alpha + l_\alpha$, $m_0 + \ldots + m_q + p_2 = m$, $\{x_k\}$ be a net of ordered nodes, $a < \ldots < x_{k-1} < x_k < x_{k+1} \ldots < b$. Further it will be considered the grid of equidistant points $x_k = a + kh$, $h > 0$. Let function u be such that $u \in C^m[a,b]$. Suppose that φ_j, $j = 1, \ldots, m$, is a Chebyshev system on $[a,b]$, in which case the functions $\varphi_j \in C^m[a,b]$, $j = 1, \ldots, m$, are nonzero on $[a,b]$. We construct

$$\widetilde{u}(x) = \sum_{\alpha=0}^{q} \sum_{j=k-l_\alpha+1}^{k+s_\alpha} u^{(\alpha)}(x_j)\, \omega_{j,\alpha}(x) +$$

$$+ \sum_{i=1}^{p_2} \left(\int_{x_{k-i}}^{x_k} u(t)dt \right) \omega_k^{<-i>}(x),$$

for approximation the function $u(x)$ on the interval $[x_k, x_{k+1}]$.

Functions $\omega_{k,\alpha}(x)$, $\omega_k^{<-i>}(x)$ are such that $supp\, \omega_{k,\alpha} = [x_{k-s_\alpha}, x_{k+l_\alpha}]$, $\alpha = 0, 1, \ldots, q$, $supp\, \omega_{k,\alpha} \subset supp\, \omega_{k,\beta}$, $\beta < \alpha$, $supp\, \omega_k^{<-i>} = [x_k, x_{k+1}]$. Functions $\omega_{k,\alpha}(x)$, $\omega_k^{<-i>}(x)$ are determined from the system of equations, which are called the approximation identities:

$$\widetilde{u}(x) = u(x), \text{ for } u(x) = \varphi_\nu(x), \ \nu = 1, \ldots, m.$$

Introduce the notations
$\Phi(x) = (\varphi_1(x), \ldots, \varphi_m(x))^T$,

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$\Phi_\alpha(x) = (\varphi_1^{(\alpha)}(x), \ldots, \varphi_m^{(\alpha)}(x))^T$,
$\Psi_{k,\alpha} = (\Phi_\alpha(x_{k-l_\alpha+1}), \ldots, \Phi_\alpha(x_{k+s_\alpha}))$,
$S\Phi_{p_2} = (\int_{x_{k-1}}^{x_k} \Phi(t)dt, \ldots, \int_{x_{k-p_2}}^{x_k} \Phi(t)dt)$.
Then the system determinant takes the form

$$\Delta = det(\Psi_{k,0}, \ldots, \Psi_{k,q}, S\Phi_{p_2}).$$

Sometimes a numerical quantity of determinant (if $h \neq const$) may be approximately equal to 0. Suppose that for the chosen values of parameters, the determinant is nonzero. Then the basis functions $\omega_{j,\alpha}(x)$, $\omega_k^{<-i>}(x)$ can be determined by Cramer's formulas. In particular, for finding the basis function $\omega_{k,\alpha}(x)$ on the interval $[x_k, x_{k+1}]$ it can be used the following relation
$\omega_{k,\alpha}(x) = det(\Psi_{k,0}, \ldots,$
$\Phi_\alpha(x_{k-l_\alpha+1}), \ldots, \Phi_\alpha(x_{k-1}), \Phi(x), \Phi_\alpha(x_{k+1}), \ldots$
$\ldots, \Phi_\alpha(x_{k+s_\alpha}) \ldots, \Psi_{k,q}, S\Phi_{p_2})/\Delta$.

Then the constructed splines $\omega_{k,\alpha}(x)$, $\omega_k^{<-i>}(x)$ and the approximation $\widetilde{u}(x)$ have the following properties:

1) at the ends of each interval $[x_k, x_{k+1}]$ we have $u^{(\alpha)}(x_k)=\widetilde{u}^{(\alpha)}(x_k)$, $u^{(\alpha)}(x_{k+1}) = \widetilde{u}^{(\alpha)}(x_{k+1})$, $\alpha = 0, 1, \ldots, q$, $\widetilde{u} \in C^q[a, b]$;

2) $\int_{x_{k-i}}^{x_k} u(t)dt = \int_{x_{k-i}}^{x_k} \widetilde{u}(t)dt$, $i = 1, \ldots, p_2$;

3) For polynomial and trigonometrical system $\{\varphi_i\}$ on equidistant set with step h we have $|\omega_k^{<-i>}(x)| \leq \widetilde{K}_i/h$, $|\omega_{k,\alpha}(x)| \leq \widetilde{C}_\alpha h^\alpha$, here $\widetilde{K}_i, \widetilde{C}_\alpha$ are certain constants.

In general we assume that a nonpolynomial system of functions $\{\varphi_i\}$ is chosen in such a way that the properties 3 are fulfilled.

3 The error of approximation

Find first the relation for $u(x)$ for computing the approximation error. Construct a homogeneous linear equation, which has a fundamental system of solutions $\varphi_1(x), \ldots, \varphi_m(x)$. Let us find the function $u(x)$ in the form convenient for obtaining error estimation. Construct first a homogeneous linear equation, having a fundamental system of equations φ_i. Let us generate next equation for $x \in [x_k, x_{k+1}] \subset [a, b]$:

$$\begin{vmatrix} \varphi_1(x), & \varphi_2(x), & \ldots & \varphi_m(x), & u(x) \\ \varphi_1'(x), & \varphi_2'(x), & \ldots & \varphi_m'(x), & u'(x) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \varphi_1^{(m)}(x), & \varphi_2^{(m)}(x), & \ldots & \varphi_m^{(m)}(x), & u^{(m)}(x) \end{vmatrix} = 0.$$

Here Wronskian

$$W(x) = \begin{vmatrix} \varphi_1(x), & \varphi_2(x), & \ldots & \varphi_m(x) \\ \varphi_1'(x), & \varphi_2'(x), & \ldots & \varphi_m'(x) \\ \ldots & \ldots & \ldots & \ldots \\ \varphi_1^{(m-1)}(x), & \varphi_2^{(m-1)}(x), & \ldots & \varphi_m^{(m-1)}(x) \end{vmatrix}$$

does not equal to zero. Expending the determinant according to the elements of the last column and dividing all terms of the obtained equation by $W(x)$, one can obtained the desired equation $Lu = u^{(m)}(x) + p_1(x)u^{(m-1)}(x) + \ldots + p_m(x)u(x) = 0$. Construct now a general solution of nonhomogeneous equation $Lu = F$ by the method of variation of the constants. Suppose,

$$u(x) = \sum_{i=1}^{m} C_i(x)\varphi_i(x).$$

Then

$$C_i(x) = \int_{x_k}^{x} \frac{W_{mi}(t)F(t)}{W(t)}dt + c_i,$$

where c_i are arbitrary constants. Since $F = Lu$, one has

$$u(x) = \sum_{i=1}^{m} \varphi_i(x) \int_{x_k}^{x} \frac{W_{mi}(t)\ Lu(t)}{W(t)}dt + \sum_{i=1}^{m} c_i\varphi_i(x).$$

where $W_{mi}(x)$ are algebraic complements of elements of i-th column of m-th row of determinant $W(x)$. Let us estimate $|r| = |\widetilde{u}(x) - u(x)|$. With help of the approximation identities we have in nonpolynomial case (see [6])

$$|r| \leq K_1 \|Lu\|_{[x_{k-l_0}, x_{k+s_0}]} h^m, \ \ K_1 > 0.$$

In polynomial case we have

$$|r| \leq K_2 \|u^{(m)}\|_{[x_{k-l_0}, x_{k+s_0}]} h^m, \ \ K_2 > 0.$$

4 Solution of the Cauchy Problem for one Equation

We shell solve the Cauchy problem

$$y' = f(x, y(x)),\ y(x_0) = y_0,\ x \in [x_0, X].$$

Consider the integral identity

$$y(x_{j+1}) = y(x_j) + \int_{x_j}^{x_{j+1}} y'(x)dx.$$

We replace $y'(x)$ by the integro-differential spline $\tilde{u}(x)$. Now we have

$$y(x_{j+1}) = y(x_j) + \int_{x_j}^{x_{j+1}} \tilde{u}(x)dx + R,$$

where $R = \int_{x_j}^{x_{j+1}}(\widetilde{u}(x) - u(x))dx$, taking into account the error of approximation by the integro-differential spline we have

$|R| \leq h^{m+1}K_3\|\ Lu\|,\ K_3 > 0.$

5 Numerical methods for $q = 0$

We have for $q = 0$ and $x \in [x_j, x_{j+1}]$

$$\widetilde{u}(x) = \sum_{k=j-l+1}^{j+s} u(x_k)\,\omega_k(x) +$$

$$+ \sum_{i=1}^{p_2} \left(\int_{x_{j-i}}^{x_j} u(t)dt \right) \omega_j^{<-i>}(x).$$

Here $\omega_{k,0}(x) = \omega_k(x)$.

5.1 Numerical method 1

Let us take $p_2 = 1$. We approximate a function $u(x)$ for $[x_j, x_{j+1}]$ by expression in a form

$$\tilde{u}(x) = u(x_j)\omega_j(x) + u(x_{j+1})\omega_{j+1}(x) +$$

$$+ \left(\int_{x_{j-1}}^{x_j} u(x)dx \right) \omega_j^{<-1>}(x).$$

Here basic splines $\omega_j(x), \omega_{j+1}(x), \omega_j^{<-1>}(x)$ we find from conditions

$$\tilde{u}(x) = u(x),\ u(x) = \varphi_1(x),\ \varphi_2(x),\ \varphi_3(x).$$

We replace integrand in

$$y(x_{j+1}) = y(x_j) + \int_{x_j}^{x_{j+1}} y'(x)dx$$

by $\tilde{u}(x)$

$$y(x_{j+1}) = y(x_j) + \int_{x_j}^{x_{j+1}} \tilde{u}(x)dx + R,$$

we obtain

$$y(x_{j+1}) = y(x_j) + u(x_j) \int_{x_j}^{x_{j+1}} \omega_j(x)dx +$$

$$+u(x_{j+1}) \int_{x_j}^{x_{j+1}} \omega_{j+1}(x)dx +$$

$$+ \int_{x_{j-1}}^{x_j} u(x)dx \int_{x_j}^{x_{j+1}} \omega_j^{<-1>}(x)dx + R.$$

Now we have the next implicit method:

$$y_{j+1} = y_j(1 + I^{<-1>}) - y_{j-1}(I^{<-1>}) +$$

$$+f(x_j, y_j)I_0 + f(x_{j+1}, y_{j+1})I_1,$$

where

$$I^{<-1>} = \int_{x_j}^{x_{j+1}} \omega_j^{<-1>}(x)dx,\ I_0 = \int_{x_j}^{x_{j+1}} \omega_j(x)dx,$$

$$I_1 = \int_{x_j}^{x_{j+1}} \omega_{j+1}(x)dx.$$

Now we construct $I^{<-1>}$, I_0, I_1 for polynomial and nonpolynomial cases.

a) Let us take $\varphi_1(x) = 1, \varphi_2(x) = e^{(x/2)}, \varphi_3(x) = e^{(-x/2)}, h = const.$

We receive easily

$$I_0 = -A_0/B_0,$$

$$A_0 = (-h/2 - 1)e^{(-h)} + (1 - h/2)e^{(h)} +$$

$$+(2 + h)e^{(-h/2)} + (h - 2)e^{(h/2)} - h,$$

$$B_0 = (-1 + h/4)e^{(-h/2)} + (-1 - h/4)e^{(h/2)} +$$

$$+\frac{1}{2}e^{(h)} + 1 + \frac{1}{2}e^{(-h)},$$

$$I_1 = A_1/B_1,$$

$$A_1 = (2 - h)e^{(-h/2)} + (-2 - h)e^{(h/2)} - e^{(-h)} +$$

$$+2h + e^{(h)},$$

$$B_1 = (-1 + h/4)e^{(-h/2)} + (-1 - h/4)e^{(h/2)} +$$

$$+\frac{1}{2}e^{(h)} + 1 + \frac{1}{2}e^{(-h)},$$

$$I^{<-1>} = -A_2/B_2,$$

$$A_2 = ((-1 - h/4)e^{(-h/2)} + (-1 + h/4)e^{(h/2)} + 2),$$

$$B_2 = (-1 + h/4)e^{(-h/2)} + (-1 - h/4)e^{(h/2)} +$$

$$+\frac{1}{2}e^{(h)} + 1 + \frac{1}{2}e^{(-h)},$$

and

$$|R| \leq\ Kh^4 \|4y^{\mathrm{IV}} - y^{\mathrm{II}}\|_{[x_{j-1}, x_{j+1}]},\ K > 0.$$

b) In case $\varphi_1(x) = 1, \varphi_2(x) = x, \varphi_3(x) = x^2$, we have

$$y_{j+1} = y_j \left(\frac{4}{5}\right) - y_{j-1} \left(\frac{-1}{5}\right) +$$

$$+f(x_j, y_j)\frac{4h}{5} + f(x_{j+1}, y_{j+1})\frac{2h}{5},$$

$$|R| \leq\ Kh^4 \|y^{\mathrm{IV}}\|_{[x_{j-1}, x_{j+1}]},\ K > 0.$$

c) In case $\varphi_1(x) = 1, \varphi_2(x) = \sin(x), \varphi_3(x) = \cos(x)$, we have

$$I_0 = 2\frac{(\cos^2(h) + h\sin(h)\cos(h) - 1)}{(2\sin(h)\cos(h) - h\cos(h) - h)},$$

$$I_1 = -2\frac{(\cos^2(h) + h\sin(h) - 1)}{(2\sin(h)\cos(h) - h\cos(h) - h)},$$

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$$I^{<-1>} = \frac{(-h\cos(h) + 2\sin(h) - h)}{(2\sin(h)\cos(h) - h\cos(h) - h)}.$$

Let us solve the problem

$$y' = -150(y - cos(x)), y(0) = 0, x \in [0, 1].$$

The exact solution is the next: $y(x) = (22500/22501)\cos(x) + (150/22501)\sin(x) - (22500/22501)\exp(-150x)$. Let us take $h = 0.001$. The error of solution of the Cauchy problem by methods b and c are represented on graphs 1 and 2.

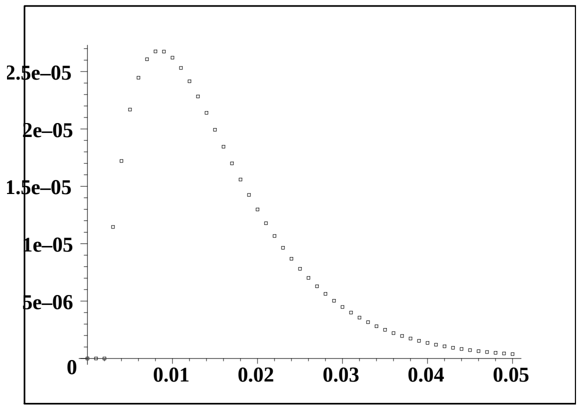

Figure 1: Graph of the error of the solution of the problem $y' = -150(y - cos(x))$ by method b

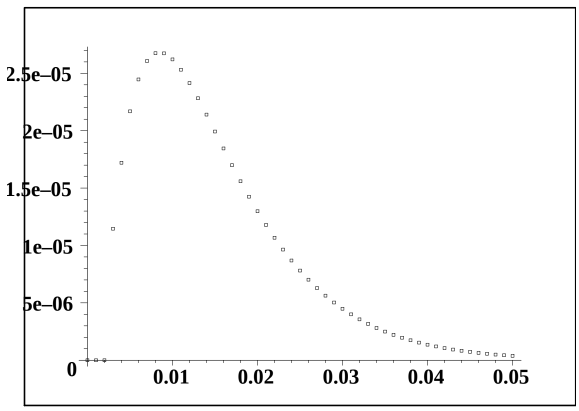

Figure 2: Graph of the error of the solution of the problem $y' = -150(y - cos(x))$ by method c

5.2 Numerical method 2

Now let us approximate function $u(x)$ by

$$\tilde{u}(x) = u(x_j)\omega_j(x) + u(x_{j+1})\omega_{j+1}(x) +$$
$$+ \left(\int_{x_{j-1}}^{x_j} u(x)dx\right) \omega_j^{<-1>}(x) +$$
$$+ \left(\int_{x_{j-2}}^{x_j} u(x)dx\right) \omega_j^{<-2>}(x)$$

on $[x_j, x_{j+1}]$. Here $\omega_j(x)$, $\omega_{j+1}(x)$, $\omega_j^{<-1>}(x)$, $\omega_j^{<-2>}(x)$ we determine from the equations:

$$\tilde{u}(x) = u(x),\ u(x) = \varphi_1(x),\ \varphi_2(x),\ \varphi_3(x),\ \varphi_4(x).$$

So we have

$$y_{j+1} = y_j\left(1 + I^{<-1>} + I^{<-2>}\right) -$$
$$-y_{j-1}I^{<-1>} - y_{j-2}I^{<-2>} +$$
$$+f(x_j, y_j)I_0 + f(x_{j+1}, y_{j+1})I_1.$$

In polynomial case $\varphi_1(x) = 1, \varphi_2(x) = x, \varphi_3(x) = x^2, \varphi_4(x) = x^3$, we have

$$I^{<-2>} = \int_{x_j}^{x_{j+1}} \omega_j^{<-2>}(x)dx = 1/17,$$

$$I^{<-1>} = \int_{x_j}^{x_{j+1}} \omega_j^{<-1>}(x)dx = -9/17,$$

$$I_0 = \int_{x_j}^{x_{j+1}} \omega_j(x)dx = 18h/17,$$

$$I_1 = \int_{x_j}^{x_{j+1}} \omega_{j+1}(x)dx = 6h/17,$$

$$y_{j+1} = y_j\left(\frac{9}{17}\right) - y_{j-1}\left(\frac{-9}{17}\right) - y_{j-2}\left(\frac{1}{17}\right) +$$
$$+f(x_j, y_j)\left(\frac{18h}{17}\right) + f(x_{j+1}, y_{j+1})\left(\frac{6h}{17}\right).$$

The error has the form:

$$|R| \leq\ Kh^5\|y^{\mathrm{V}}\|_{[x_{j-2}, x_{j+1}]}, K > 0.$$

Let us solve the problem

$$y' = -150(y - \cos(x)), y(0) = 0, x \in [0, 1].$$

The errors of the solution of the Cauchy problem by method 2 are represented on graphs 3, 4.

Now let us solve next problem
$y' = -2(y - \sin(x)) + \cos(x), y(0) = 0.$

The exact solution is $y = sin(x)$. The error of solution of the Cauchy problem by same method is represented on graph 5.

6 Conclusion

The results explained in the previous sections show that the numerical methods could be used on practical calculations.

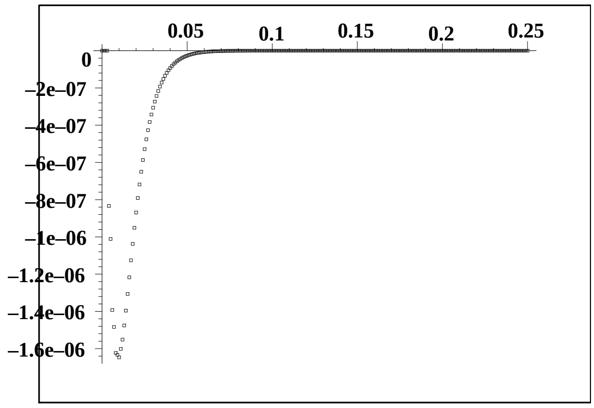

Figure 3: Graph of the error of the solution of the problem $y' = -150(y - \cos(x))$ ($h = 0.001$)

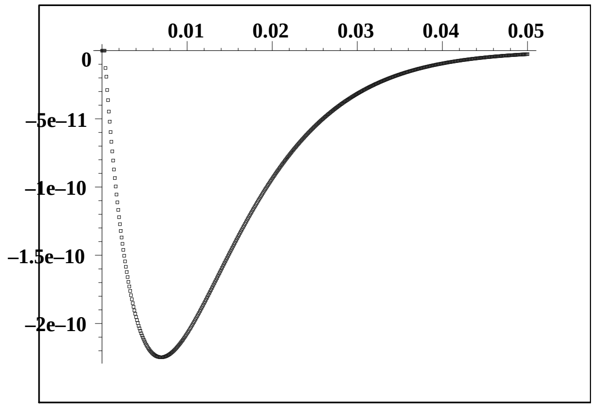

Figure 4: Graph of the error of the solution of the problem $y' = -150(y - \cos(x))$ ($h = 0.0001$)

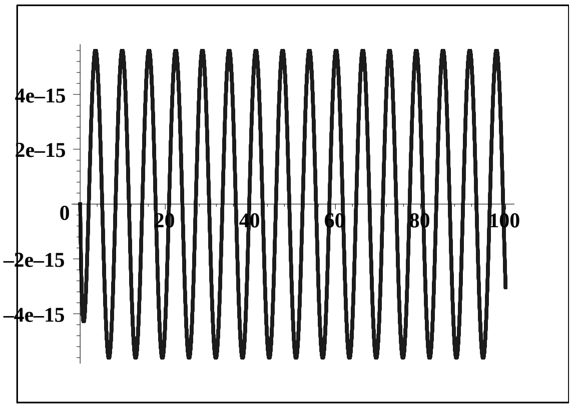

Figure 5: Graph of the error of the solution of the problem $y' = -2(y - \sin(x)) + \cos(x)$, $y(0) = 0$ ($h = 0.001$)

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