A new theorem on the localization of factored Fourier Series

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Abstract: In this paper, a general theorem dealing with the local property of \( |A, p_n|_k \) summability of factored Fourier series has been proved, which generalizes some known results. This new theorem also includes several new results.

Key Words: Summability factors, absolute matrix summability, Fourier series, infinite series, Hölder inequality, Minkowski inequality

1 Introduction

Let \( \sum a_n \) be a given infinite series with partial sums \( (s_n) \). Let \( A = (a_{nm}) \) be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then \( A \) defines the sequence-to-sequence transformation, mapping the sequence \( s = (s_n) \) to \( As = (A_n(s)) \), where

\[
A_n(s) = \sum_{v=0}^{n} a_{nv}s_v, \quad n = 0, 1, \ldots
\]

The series \( \sum a_n \) is said to be summable \( |A|_k \), \( k \geq 1 \), if \[23\]

\[
\sum_{n=1}^{\infty} n^{k-1} |\Delta A_n(s)|^k < \infty,
\]

where

\[
\Delta A_n(s) = A_n(s) - A_{n-1}(s).
\]

Let \((p_n)\) be a sequence of positive numbers such that

\[
P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty,
\]

\[
(P_{-i} = p_{-i} = 0, \quad i \geq 1).
\]

The sequence-to-sequence transformation

\[
t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v
\]

defines the sequence \((t_n)\) of the Riesz mean or simply the \((\tilde{N}, p_n)\) means of the sequence \((s_n)\) generated by the sequence of coefficients \((p_n)\) \[12\].

The series \( \sum a_n \) is said to be summable \( |\tilde{N}, p_n|_k \), \( k \geq 1 \), if \[3\]

\[
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty.
\]

In the special case, when \( p_n = 1 \) for all values of \( n \) (resp. \( k = 1 \)), \( |\tilde{N}, p_n|_k \) summability is the same as \( |C, 1|_k \) (resp. \( |\tilde{N}, p_n|_k \)) summability. Also if we take \( k = 1 \) and \( p_n = 1/n + 1 \), summability \( |\tilde{N}, p_n|_k \) is equivalent to the summability \( |R, \log n, 1| \).

It is said to be summable \( |A, p_n|_k \), \( k \geq 1 \), if \[22\]

\[
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta A_n(s)|^k < \infty
\]

where

\[
\Delta A_n(s) = A_n(s) - A_{n-1}(s).
\]

Let \((\varphi_n)\) be any sequence of positive real numbers. The series \( \sum a_n \) is said to be summable \( \varphi - |A, p_n|_k \), \( k \geq 1 \), if \[19\]

\[
\sum_{n=1}^{\infty} \varphi_n^{k-1} |\Delta A_n(s)|^k < \infty.
\]

2 Known Result

Theorem 1 Let \( k \geq 1 \). If \((\lambda_n)\) is a non-negative and non-increasing sequence such that \( \sum p_n\lambda_n \) is convergent, then the summability \( |\tilde{N}, p_n|_k \) of the series \( \sum C_n(t)\lambda_n P_n \) at a point is a local property of the generating function \( f \).
3 Main Theorem

Let $k \geq 1$. If $A = (a_{nv})$ is a positive normal matrix such that
\[ a_{n0} = 1 \quad n = 0, 1, \ldots, \]
\[ a_{n-1,v} \geq a_{nv} \quad for \quad n \geq v + 1 \]
\[ a_{nn} = O\left(\frac{p_n}{P_n}\right) \] (11)
and $\left(\frac{\varphi_m p_m}{P_m}\right)$ be a non-increasing sequence. If all the conditions of Theorem 1 are satisfied and $(\varphi_n)$ is any sequence of positive constants such that
\[ \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} p_v \lambda_v = O(1) \quad as \quad m \to \infty \] (12)
\[ \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} P_v \Delta \lambda_v = O(1) \quad as \quad m \to \infty \] (13)
then the summability $\varphi - |A, p_n|_k$ of the series
\[ \sum C_n(t) P_n \lambda_n \] at a point is a local property of the generating function $f$.

We need the following lemmas for the proof of our theorem.

Lemma 1 [8] If $(\lambda_n)$ is a non-negative and non-increasing sequence such that $\sum p_n \lambda_n$ is convergent, where $(p_n)$ is a sequence of positive numbers such that $P_n \to \infty$ as $n \to \infty$, then $P_n \lambda_n = O(1)$ as $n \to \infty$ and $\sum P_n \Delta \lambda_n < \infty$.

Lemma 2 Let $k \geq 1$ and $(s_n) = a_1 + a_2 + \ldots + a_n = O(1)$. If $(\lambda_n)$ is a non-negative and non-increasing sequence such that $\sum p_n \lambda_n$ is convergent and the conditions (9-13) are satisfied, then the series $\sum a_n \lambda_n P_n$ is summable $\varphi - |A, p_n|_k$.

4 Conclusion

The behaviour of the Fourier series at $t = x$ is a local property of $f$ (i.e., it depends only on the behaviour of $f$ in an arbitrarily small neighbourhood of $x$), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of $f$. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of $x$ depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of Main Theorem is a consequence of Lemma 2.

1. If we take $\varphi_n = \frac{p_n}{P_n}$, then we get a theorem dealing with $|A, p_n|_k$ summability.
2. If we take $\varphi_n = \frac{p_n}{P_n}$ and $a_{nv} = \frac{p_n}{P_n}$, then we get Theorem 1.
3. If we take $a_{nv} = \frac{p_n}{P_n}$, then we have another result dealing with $\varphi - |N, p_n|_k$ summability.
4. If we take $a_{nv} = \frac{p_n}{P_n}$ and $p_n = 1$ for all values of $n$, then we get a result for dealing with $\varphi - |C, 1|_k$ summability [20].
5. If we take $\varphi_n = n, a_{nv} = \frac{p_n}{P_n}$ and $p_n = 1$ for all values of $n$, then we get a result for $|C, 1|_k$ summability [11].

References:


