On the Pole Assignment for Linear Time Invariant (LTI), Discrete Time and Continuous Time Multiple Inputs Multiple Outputs (MIMO) Systems via Output Feedback Control.

NIKOS E. MASTORAKIS
Military Institutions of University Education (MIUE), Hellenic Naval Academy,
Chair of Computer Science, Terma Hatzikyriakou, 18539, Piraeus, GREECE

Abstract: - In this brief, the problem of the pole assignment of a linear time invariant (*LTI*), discrete/continuous multiple inputs/multiple outputs (*MIMO*) time system via output feedback is considered. The problem is reduced to the minimization of the distance of the desired characteristic polynomial from the characteristic polynomial of the under output feedback control law system at the unit circle. Numerical examples are given. The method is easily extended in 2-D systems.

Keywords: - Control Theory, Pole Assignment, Feedback Control, Minimization Techniques.
CSCC'99 Proceedings:-Pages 1001-1006

1 Introduction

Consider the LTI Discrete System described by:

$$\mathbf{x}(n+1) = \mathbf{A}\mathbf{x}(n) + \mathbf{B}\mathbf{u}(n) \qquad (1.1)$$

$$\mathbf{y}(n) = \mathbf{C}\mathbf{x}(n) + \mathbf{D}\mathbf{u}(n) \qquad (1.2)$$

where $\mathbf{u}(n) \in \Re^{\nu_1}$ is the input vector of the system, $\mathbf{y}(n) \in \Re^{\nu_2}$ is the output vector, $\mathbf{x}(n) \in \Re^{\nu}$ is the state vector and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are matrices of appropriate dimensions. In Continuous Systems a similar analysis yields identical results.

Consider the following feedback control law:

$$\mathbf{u}(n) = \mathbf{F}\mathbf{y}(n) + \mathbf{v}(n) \qquad (2)$$

Then the arbitrary eigenvalue assignment problem with output feedback consists of the design of a regulator of the form (2) in order to:

$$det\left[\mathbf{z}\mathbf{I} - \left(\mathbf{A} + \mathbf{B}\mathbf{F}\mathbf{C}(1 - \mathbf{F}\mathbf{D})^{-1}\right)\right] = p(z) \qquad (3)$$

where $p(z)$ is an arbitrary monic polynomial ν-degree in $\Re[z]$ where $\Re[z]$ denotes the ring of polynomials with real coefficients. It is noted that the placement of the poles of the closed-loop system becomes with the help of $\nu_1 * \nu_2$ parameters of the regulator $\mathbf{F}$.

In 1970, Davison, [1], and Davison and Chow, [2], proved that $max(\nu_1, \nu_2)$ eigenvalues of the matrix $[\mathbf{Iz} - \mathbf{A} - \mathbf{BFC}]$ can be placed in arbitrarly selected positions. Moreover, Davison and Wang, [3], and Kimura, [4], showed independently that "almost all" the systems (1.1) and (1.2) with $\nu_1 + \nu_2 \geq \nu + 1$ have arbitrary pole assignment. An alternative proof of the above result was given from Brockett and Byrnes, [5], and Schumacher, [6]. Herman and Martin, [7], proved that almost all the systems (1.1) and (1.2) with $\nu_1 * \nu_2 \geq \nu$ have arbitrary pole assignment with complex numbers control law of the form (2). Willems and Hesselink, [8], proved that the condition $\nu_1 * \nu_2 \geq \nu$ is necessary but no sufficient for the solution of the problem in the set of the real numbers. Karcanias and Giannakopoulos, [9], gave a necessary condition for the solution of the above problem in the set of the real numbers concerning the degree of the Plucker's matrix. Wang, [10], showed that if the system (1.1), (1.2) is generic and $\nu_1 * \nu_2 \geq \nu$, then all the poles of it can be placed in arbitrarly selected positions. An alternative sufficient condition was also given by Wang, [11]. His results concern a very limited category of systems and furthermore they are not accompanied from a certain algorithm which places the poles in the desired position. Alternative proofs of the Wang's result, [10], were given from Karcanias and Leventides, [12], and from Rosenthal, Willems and Schumacher, [13]. A complete reference is given in the review papers of Syrmos and Dorato, [14], Kimura, [15], and Byrnes, [16]. Recent results can also be found in [17], and, [18].

2 Problem Formulation

Consider the following LTI System :

$$\mathbf{x}(n+1) = \mathbf{A}\mathbf{x}(n) + \mathbf{B}\mathbf{u}(n) \tag{4.1}$$

$$\mathbf{y}(n) = \mathbf{C}\mathbf{x}(n) + \mathbf{D}\mathbf{u}(n) \tag{4.2}$$

where $\mathbf{u}(n)$ is the input vector, $\mathbf{y}(n)$ is the output vector, $\mathbf{x}(n)$ is the state-space vector, $\mathbf{A}$ is a $\nu x \nu$ matrix, $\mathbf{B}$ is a $\nu x \nu_1$ matrix, $\mathbf{C}$ is a $\nu_2 x \nu$ matrix and $\mathbf{D}$ is a $\nu_2 x \nu_1$ matrix. It is well-known that the characteristic polynomial of the above system (4.1), (4.2) is:

$$a(z) = det(\mathbf{zI} - \mathbf{A}) \tag{5}$$

The problem in question is how under the output feedback control law:

$$\mathbf{u}(n) = \mathbf{F}\mathbf{y}(n) + \mathbf{v}(n) \tag{6}$$

the characteristic polynomial (of ν-degree) of the original system (4.1), (4.2) will be led to the desired polynomial $p(z)$, where $p(z)$ is an arbitrary ν-degree monic polynomial of $\Re[z]$.

Obviously substituting (6) into (4.1), (4.2) one finds:

$$\begin{aligned}\mathbf{x}(n+1) = &\left(\mathbf{A} + \mathbf{BF}(\mathbf{I} - \mathbf{DF})^{-1}\mathbf{C}\right)\mathbf{x}(n) + \\ &+ \left(\left(\mathbf{BF}(\mathbf{I} - \mathbf{DF})^{-1}\mathbf{D} + \mathbf{BF}\right)\right)\mathbf{v}(n)\end{aligned} \tag{7.1}$$

$$\mathbf{y}(n) = (\mathbf{I} - \mathbf{DF})^{-1}\mathbf{C}\mathbf{x}(n) + (\mathbf{I} - \mathbf{DF})^{-1}\mathbf{D}\mathbf{v}(n) \tag{7.2}$$

Therefore the characteristic polynomial under output feedback control law will be:

$$a_f(z) = det\left(\mathbf{zI} - \left(\mathbf{A} + \mathbf{BF}(\mathbf{I} - \mathbf{DF})^{-1}\mathbf{C}\right)\right) \tag{8}$$

The problem is: which $\mathbf{F}$ must be selected in order to have:

$$a_f(z) = p(z) \tag{9}$$

In the relevant literature, there exists a great many number of theoretical results for the solution of (9) with respect to $\mathbf{F}$. Unfortunately, all these theoretical results are not appropriate for practical purposes at all. For this reason, in the present study, an optimization criterion is proposed for the approximate validity of (9). The proposed criterion is:

$$min \frac{1}{2\pi j} \oint_{|z|=1} \left| p(z) - a_f(z) \right|^{2\rho} \frac{dz}{z} \tag{10}$$

where ρ is an arbitrary positive integer, or equivalently:

$$min \frac{1}{2\pi j} \oint_{|z|=1} \left| p(z) - det\left(\mathbf{zI} - \left(\mathbf{A} + \mathbf{BF}(\mathbf{I} - \mathbf{DF})^{-1}\mathbf{C}\right)\right) \right|^{2\rho} \frac{dz}{z} \tag{11}$$

where the selection $\rho = 1$ is usually used.

From (11), with $\rho = 1$ after elementery manipulation, one finds:

$$min \sum_{k=0}^{\nu} \frac{1}{k!} \frac{\partial^k}{\partial z^k} \left(p(z) - det\left(\mathbf{zI} - \left(\mathbf{A} + \mathbf{BFC}(\mathbf{I} - \mathbf{DF})^{-1}\right)\right)\right)^2 \tag{12}$$

The above minimization problem is solved via a variety of numerical techniques. In this paper, the Levenberg-Marquardt routine for solving nonlinear least squares problems is used, [19]÷[22]. The problem for this routine, is stated as follows:

$$\begin{aligned} &minimize_{over\ \underline{x}} \left\{ f_1^2(\underline{x}) + \ldots + f_m^2(\underline{x}) \right\} \\ &where\ \underline{x} = \left\{ x_1, \ldots, x_N \right\} \end{aligned} \tag{13}$$

A sequence of approximation to the minimum point is generated by:

$$\underline{x}^{n+1} = \underline{x}^{n} - \left[a_n \mathbf{D}_n + \mathbf{J}_n^T \mathbf{J}_n\right]^{-1} \cdot \mathbf{J}_n^T \underline{f}\left(\underline{x}^n\right) \quad (14)$$

where $\mathbf{J}_n$ is the mathematical Jacobian matrix evaluated at $\underline{x}^n$. $\mathbf{D}_n$ is a diagonal matrix equal to the diagonal of $\mathbf{J}_n^T \mathbf{J}_n$ and a_n is a positive scaling constant.

Example:- If we suppose that we have:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 0 \\ -1 & -1 & 1 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \text{ and } \mathbf{F} = \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}$$

then after the appropriate calculations (using a simple Mathematica code) we find that $f_1 = 1.9655$, $f_2 = 2.0076$, $f_3 = 2.2952$ and $f_4 = 1.1539$ Therefore:

$$\mathbf{F} = \begin{bmatrix} 1.9655 & 2.0076 \\ 2.2951 & 1.1539 \end{bmatrix}$$

The Mathematica code is as follows:

```
a=.;b=.;f=.;

i2={{1,0},{0,1}};
i3={{1,0,0},{0,1,0},{0,0,1}};
a={{1,2,-1},{-3,4,0},{-1,-1,1}};
b={{1/2,-1/2},{-1/2,1/2},{1,0}};
c={{1,2,1},{0,1,1}};
d={{1,-1},{-1,0}};
f={{f1,f2},{f3,f4}};

inv=Inverse[i2-d.f];
difference=(z+1)(z+2)(z+3)-Det[z i3-(a+b.f.w.c)];
z=x;
coef1=difference/.x->0;
coef2=D[difference,x]/.x->0;
coef3=(D[difference,{x,2}])/2/.x->0;
coef4=(D[difference,{x,3}])/6/.x->0;
scoef=coef1 coef1+coef2 coef2+coef3 coef3+coef4
coef4;
simp=Simplify[scoef];
FindMinimum[simp,{f1,-.5},{f2,1},{f3,.5},{f4,1},
MaxIterations->100]
```

3 Extension in 2-D Discrete Systems

In 2-D Discrete Systems, quite analogously, we have:

$$\mathbf{x}(n+1) = \mathbf{A}\mathbf{x}(n) + \mathbf{B}\mathbf{u}(n) \quad (15.1)$$

$$\mathbf{y}(n) = \mathbf{C}\mathbf{x}(n) + \mathbf{D}\mathbf{u}(n) \quad (15.2)$$

where the following compact notation is used:

$$\mathbf{x}(n) = \begin{bmatrix} \mathbf{x}^{(h)}(n_1,n_2) \\ \mathbf{x}^{(v)}(n_1,n_2) \end{bmatrix},$$

$$\mathbf{x}(n+1) = \begin{bmatrix} \mathbf{x}^{(h)}(n_1+1,n_2) \\ \mathbf{x}^{(v)}(n_1,n_2+1) \end{bmatrix},$$

$$\mathbf{u}(n) = \mathbf{u}(n_1,n_2),$$

$$\mathbf{y}(n) = \mathbf{y}(n_1,n_2)$$

where $\mathbf{x}^{(h)}(n_1,n_2)$ and $\mathbf{x}^{(v)}(n_1,n_2)$ are the orizontal and vertical state space vectors and $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ are matrices of appropriate dimensions. The above model (15.1), (15.2) is known as the Roesser model. The characteristic polynomial of (15.1), (15.2) is:

$$a(z_1,z_2) = det\left(\begin{bmatrix} z_1\mathbf{I}^{(h)} & 0 \\ 0 & z_2\mathbf{I}^{(v)} \end{bmatrix} - \mathbf{A}\right) \quad (16)$$

The problem in question is how under the output feedback:

$$\mathbf{u}(n_1,n_2) = \mathbf{F}\mathbf{y}(n_1,n_2) + \mathbf{v}(n_1,n_2) \qquad (17)$$

the characteristic polynomial of ν_1,ν_2-degree of the original system (15.1), (15.2), will be led to the desired polynomial $p(z_1,z_2)$ where $p(z_1,z_2)$ is an arbitrary ν_1,ν_2-degree monic polynimial of $\Re[z_1,z_2]$.

Obviously substituting (17) into (15.1), (15.2) and using the above compact notation one finds:

$$\begin{aligned}\mathbf{x}(n+1) &= \left(\mathbf{A}+\mathbf{BF}(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{C}\right)\mathbf{x}(n) + \\ &\quad + \left(\left(\mathbf{BF}(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{D}+\mathbf{BF}\right)\right)\mathbf{v}(n)\end{aligned} \qquad (18.1)$$

$$\mathbf{y}(n) = (\mathbf{I}-\mathbf{DF})^{-1}\mathbf{C}\mathbf{x}(n) + (\mathbf{I}-\mathbf{DF})^{-1}\mathbf{D}\mathbf{v}(n) \qquad (18.2)$$

Therefore the characteristic polynomial under output feedback control law will be:

$$a_f(z_1,z_2) = det\left(\begin{bmatrix} z_1\mathbf{I}^{(h)} & 0 \\ 0 & z_2\mathbf{I}^{(v)}\end{bmatrix} - \left(\mathbf{A}+\mathbf{BF}(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{C}\right)\right) \qquad (19)$$

where $\mathbf{I}^{(h)}$, $\mathbf{I}^{(v)}$ are identity matrices corresponding to $\mathbf{x}^{(h)}(n)$, $\mathbf{x}^{(v)}(n)$.

Our problem is: which $\mathbf{F}$ must be selected in order to have:

$$a_f(z_1,z_2) = p(z_1,z_2) \qquad (20)$$

Some theoritical results exist for the solution of (20) with respect to $\mathbf{F}$ in the recent literature. Unfortunately, these results are not appropriate for practical usage. For this reason, in the present study, the following optimization criterion will be used:

$$min \frac{1}{(2\pi j)^2} \oint_{|z_1|=1} \oint_{|z_2|=1} \left|p(z_1,z_2) - a_f(z_1,z_2)\right|^{2\rho} \frac{dz_1}{z_1}\frac{dz_2}{z_2} \qquad (21)$$

where ρ is an arbitrary positive integer, or equivalently, for $\rho = 1$, one has:

$$min \frac{1}{(2\pi j)^2} \oint_{|z_1|=1} \oint_{|z_2|=1} \left|p(z_1,z_2) - det\left(\begin{bmatrix} z_1\mathbf{I}^h & 0 \\ 0 & z_2\mathbf{I}^v\end{bmatrix} - \left(\mathbf{A}+\mathbf{BF}(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{C}\right)\right)\right|^{2\rho} \frac{dz_1}{z_1}\frac{dz_2}{z_2} \qquad (22)$$

From (22) with $\rho = 1$ after elementary manipulation, one finds:

$$min \sum_{k_1=0}^{\nu_1}\sum_{k_2=0}^{\nu_2} \left(\frac{1}{k_1!}\frac{1}{k_2!}\frac{\partial^{k_1}}{\partial z_1^{k_1}}\frac{\partial^{k_2}}{\partial z_2^{k_2}}\left(p(z_1,z_2) - det\left(\begin{bmatrix} z_1\mathbf{I}^h & 0 \\ 0 & z_2\mathbf{I}^v\end{bmatrix} - \left(\mathbf{A}+\mathbf{BF}(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{C}\right)\right)\right)\right)^2 \qquad (23)$$

The above minimization problem is also solved via the Levenberg-Marquardt routine for solving nonlinear least squares problems. An illustrative example is given.

Example: -We consider that:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 0 \\ -1 & -1 & 1 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix},$$

and $\mathbf{F} = \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}$

Also suppose that

$$\mathbf{I}^{(h)} = [1]$$

and

$$\mathbf{I}^{(v)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

After the necessary calculations one can find: $f_1 = 0.0813$, $f_2 = -0.3211$, $f_3 = 0.0033$ and $f_4 = 0.3892$. Therefore:

$$\mathbf{F} = \begin{bmatrix} 0.0813 & -0.3211 \\ 0.0033 & 0.3892 \end{bmatrix}$$

For the above result the following program written in Mathematica 2.2 for Windows was used:

```
a={{1,2,-1},{-3,4,0},{-1,-1,1}};
b={{1/2,-1/2},{-1/2,1/2},{1,0}};
c={{1,2,1},{0,1,1}};
d={{1,-1},{-1,0}};
f={{f1,f2},{f3,f4}};
z={{z1,0,0},{0,z2,0},{0,0,z2}};
i2={{1,0},{0,1}};
i3={{1,0,0},{0,1,0},{0,0,1}};

inv=Inverse[i2-d.f];
difference=(7+3 z1+9 z2-5 z1 z2-z2 z2 + z1 z2 z2)-
Det[z-(a+b.f.w.c)];
coef00=difference/.{z1->0, z2->0};
coef01=D[difference,z2]/.{z1->0, z2->0};
coef10=D[difference,z1]/.{z1->0, z2->0};
coef11=D[difference,z1,z2]/.{z1->0, z2->0};
coef02=D[difference,z2,z2]/2/.{z1->0, z2->0};
coef20=D[difference,z1,z1]/2/.{z1->0, z2->0};
coef12=D[difference,z1,z2,z2]/2/.{z1->0, z2->0};
coef21=D[difference,z1,z1,z2]/2/.{z1->0, z2->0};
coef22=D[difference,z1,z1,z2,z2]/4/.{z1->0, z2-
>0};
scoef1=coef00 coef00 +coef01 coef01 +coef10
coef10 +coef11 coef11;
scoef2=scoef1+coef02 coef02 +coef20 coef20
+coef12 coef12+coef21 coef21 +coef22 coef22;
sim=Simplify[scoef2];
FindMinimum[y,{f1,0},{f2,0},{f3,0},{f4,0},
MaxIterations->100]
```

4 Conclusion

The methodology of the (approximate) pole assignment can be proved useful and powerful tool for many practical design problems of classical automatic control. Here, this methodology was adopted for linear time invariant, discrete time systems. The analysis can be the same for continuous time systems. Also, this methodology is applied for two dimensional systems.

References:

[1] E.Davison, "On the pole assignment in linear systems with incomplete state feedback", IEEE Trans. Autom Contr. pp.348-351, 1970.

[2] E. Davison and S. Chow, " An algorithm for the assignment of closed-loop poles using output feedback in large linear multivariable systems", IEEE Trans. Autom Contr, pp.74-75, 1970.

[3] E. Davison and S. Wang "On pole assignment of in linear multivariable systems using output feedback", IEEE Trans. Autom Contr, pp.516-518, 1975.

[4] H. Kimura, "Pole assignment by gain output feedback", IEEE Trans. Autom Contr, pp.509-516, 1975.1970.

[5] R. Brockett and C. Byrnes, "Multivariable Nyquist criteria, root loci and pole placement: a geometric viewpoint", IEEE Trans. Autom Contr, pp.271-284, 1981.

[6] J. M. Schumacher, "Compesator synthesis using (C, A, B-Pairs)", IEEE Trans. Autom Contr, pp.1133-1138, 1980.

[7] R. Hermann and C. F. Martin, "Applications of algebraic geometry to linear system theory", IEEE Trans. Autom Contr, pp.19-25, 1977.

[8] J. C. Willems and W. H. Hesselink, "Generic properties of the pole placement problems", Proc. of the 7th IFAC Congress, pp.1725-1729, 1978.

[9] G. Giannakopoulos and N. Karcanias, "Pole assignment of strictly proper and proper linear systems by constant output feedback", Int. Jour. Contr, pp.543-565, 1985.

[10] X. Wang, "Pole placement by static utput feedback", J. Math. Syst. Estimation and Control, pp205-218, 1992.

[11] X. Wang, "Grassmannian, central projection and output feedback pole assignment of linear systems", IEEE Trans. Autom Contr, pp.786-7594 1996.
[12] J. Leventides and N. Karcanias, "Global aymptotic linearisation of the pole placement map: a closed-form solution for the constant output feddback problem", Automatica, No9, pp.1303-1309, 1993.
[13] J. Rosenthal, J. M. Schumacher and J. C. Willems, "Generic eigenvalue assignment by memoryless rael output feedback", Syst. Contr. Lett., 26, pp253-260, 1995.
[14] V. L. Syrmos, C. Abdalah and P. Dorato, "Static output feedback: A survey", 33rd IEEE Conf. on Dec. Contr, pp.837-842, Dec.1994.
[15] H.Kimura, "Pole assignment by output feedback: A Longstanding open problem", 33rd IEEE Conf. on Dec. Contr, pp.2101-2105, Dec.1994.
[16] C. I. Byrnes, "Pole assignment by output feedback", in Three Decades of Mathematical System Theory, H. Nijmeijer and J. M. Schumacher (eds.) pp.31-8, 1989.
[17] N. Mastorakis(ed.), "Recent Advances in Circuits and Systems", World Scientific Pu. Co. 1998.
[18] K. Kiritsis, "Contribution on the Linear Systems Control by output feedback control", Ph. D. Thesis, National Technical Univ. of Athens, 1997.
[19] Brown, K. M.(1972) Computer Oriented Methods for fitting tabular data in the linear and nonlinear least squares sense. Department of Computer, Information and Control Sciences, TR No.72-13,University of Minnesota.
[20] Brown, K. M. and Dennis, J. E. (1972) Derivate free analogues of the Levenberg-Marquardt and Gauss algorithms for nonlinear least squares approximations. Numerische Mathematik, 18, 289-297.
[21] Levenderg, K. (1944) A method for the solution of certain nonlinear problems in least squares. Quart. Appl. Meth., 2, 164-168.
[22] Marquardt, D.W. (1963) An Algorithm for least-squares estimation of non-linear parameters. J. Siam, 11,2.