Study of Stability of Multidimensional Systems using Genetic Algorithms

NIKOS E. MASTORAKIS and IOANNIS F. GONOS(*)
Military Institutions of University Education, Hellenic Naval Academy,
Chair of Computer Science, Terma Hatzikyriakou, 18539, Piraeus, GREECE
(*) also with the National Technical University of Athens, Electric Power Division, Department of Electrical Eng. and Comp. Sci. , Patission 42, 10682, Athens, GREECE.

Tel-Fax: +301 7775660

Abstract: The study of the Stability of *m*-dimensional systems is a difficult problem especially when $m \geq 3$. There exist only a few results and, unfortunately, there does not exist any practical criterion. In this brief, the stability of an *m*-dimensional system is dealt as a minimization problem of the absolute value of its characteristic polynomial over the boundaries of its variables (i.e. on the *m* unit circles). In this minimization, we seek for a global minimum. It is known that all the numerical algorithms and all the artificial neural networks' techniques can not guarantee the convergence to the total (global) minimum. On the contrary, genetic algorithms provide us the advantage of the convergence to the global minimum without the requirement of the differentiability nor of the objective function neither of the constraints. So, the problem of the stability of an *m*-D (multidimensional) system is reduced to a minimization problem of the absolute value of its characteristic polynomial over the boundaries of its variables which is solved via an appropriate genetic algorithm (GA). Numerical examples are presented. *CSCC'99 Proceedings:* - Pages 1021-1028

I. Introduction

In the study of Systems Theory, stability plays an important role, since every designed system ought to be stable. An 1-D (one-dimensional) discrete-time system is stable (in the Bounded Input - Bounded Output sense) if and only if its characteristic polynomial has not any root inside the unit disk and it has not any multiple root on the unit circle. In the system theory's literature, this kind of stability is also known as Schur Stability. Also, for practical purposes and applications, there exist many criteria like Jury's test, Hurwitz's test etc that check the stability without finding the roots of the characteristic polynomial.

An *m*-D (multidimensional) linear, shift invariant, discrete variables, system described by the transfer function $G(z_1,\ldots,z_m) = \dfrac{A(z_1,\ldots,z_m)}{B(z_1,\ldots,z_m)}$, is stable (in the Bounded Input - Bounded Output sense) if and only if

$B(0,\ldots,0,z_m) \neq 0$ for $|z_m| \leq 1$

$B(0,\ldots,0,z_{m-1},z_m) \neq 0$ for $|z_{m-1}| \leq 1,\ |z_m| = 1$

$\vdots$

$B(0,z_2,\ldots,z_{m-1},z_m) \neq 0$ for $|z_2| \leq 1,\ |z_3| = \ldots = |z_m| = 1$

$B(z_1,z_2,\ldots,z_m) \neq 0$ for $|z_1| \leq 1,\ |z_2| = \ldots = |z_m| = 1$

and $A(z_1,z_2,\ldots,z_m)$, $B(z_1,z_2,\ldots,z_m)$ have not any nonessential singularity of the second kind (i.e. $\neg\exists(z_1^*,z_2^*,\ldots z_m^*)$ with $|z_i^*| < 1,\ i = 1,\ldots,m$ (*m*>1) such as $A(z_1^*,z_2^*,\ldots z_m^*) = B(z_1^*,z_2^*,\ldots z_m^*) = 0$) ■

The above theorem is known as the theorem of *Anderson and Jury*, [1÷3].

Unfortunately, for practical purposes (filtering, design of *m*-D filters etc) we need more "convenient", more "practical" tests than the above theorem. In 2-D systems, a great variety of practical tests has been produced in the last three decades (Jury's 2-D test [1,2], Schur-Cohn test [1,2], Inners' test [12], Zeheb-Walach test [7,8], Mastorakis-Barnett test [23,25,27], Partial Energies'test [21], etc). There are also a variety of special results and other considerations [22÷30].

In *m*-D systems (*m*>3), unfortunately, we have a complete lack of such tests, though we must refer to the contributions of [3÷15].

So, it is difficult to check if a given *m*-D polynomial $B\left(z_1 \ldots, z_m\right)$ corresponds to the characteristic polynomial of a *stable* *m*-D system when *m*>2.

In the sequel, by the term stable or unstable polynomial, we will mean the characteristic polynomial of stable or unstable *m*-D (linear, shift-invariant, discrete variables) system.

An important result in the stability of *m*-D system is given by the following theorem, known as DeCarlo-Strintzis Theorem [1,2,5].

DeCarlo-Strintzis Theorem: $B\left(z_1, \ldots z_m\right)$ is a stable polynomial if and only if

$$B\left(z_1, 1, \ldots 1\right) \neq 0 \quad \text{for } \left|z_1\right| \leq 1 \qquad (1.1)$$

$$B\left(1, z_2, 1, \ldots, 1\right) \neq 0 \quad \text{for } \left|z_2\right| \leq 1 \qquad (1.2)$$

$$\vdots \qquad .$$

$$B\left(1, \ldots 1, z_m\right) \neq 0 \quad \text{for } \left|z_m\right| \leq 1 \qquad (1,m)$$

and

$$B\left(z_1, \ldots, z_{m-1}, z_m\right) \neq 0 \text{ for } \left|z_1\right| = \ldots = \left|z_m\right| = 1 \qquad (1,m+1)$$

and numerator of the transfer function and $B\left(z_1, \ldots z_m\right)$ have not any nonessential singularity of the second kind. ■

In the sequel, we always will assume that the condition of the non-existence of nonessential singularities of the second kind is fulfilled. The *m* first conditions of DeCarlo-Strintzis Theorem actually consist *m* one-dimensional conditions and are easy to be checked via any 1-D test (for example one-dimensional Jury's test).

In this brief, Genetic Algorithms (GA's) methodology is proposed in order to check the last equation of DeCarlo-Strintzis' Theorem. This methodology is presented in the next section.

II. A Genetic Algorithm for checking the *m*-dimensional systems stability

According the DeCarlo-Strintzis Theorem, the first *m* conditions can be examined via any one-dimensional test (criterion). If some of these conditions is/are not satisfied, we easily conclude that the system is unstable, without examing the last condition (1,m+1). However, if these *m* conditions are fulfilled, then Condition (1,m+1) is that which will "decide" about the stability. If it is satisfied the system is stable, otherwise is unstable. In order to investigate (1,m+1), we consider the minimum of the function *f*, where $f = f\left(w_1, w_2, \ldots, w_m\right) = \left|B\left(e^{jw_1}, e^{jw_2}, \ldots e^{jw_n}\right)\right|$. So, assume that

$$\underline{M} = \mathit{min}\ f \quad \text{over } w_i \;\; 0 \leq w_i \leq 2\pi \qquad (2)$$

Therefore (1,m+1) condition is equivalent to

$$\underline{M} > 0 \qquad (3)$$

If $\underline{M}=0$, the polynomial $B\left(z_1, \ldots z_m\right)$ is unstable. The problem with the existing methods of minimum research (numerical or neural networks' techniques) is that they usually give only local minima whereas we wish to find the global minimum over the boundaries $\left|z_1\right| = \ldots = \left|z_m\right| = 1$ (or equivalently over the set $\{w_i / 0 \leq w_i \leq 2\pi\}$)

So, we are obliged to develop a reliable and efficient global optimization technique. To this end, in this paper, a simple Genetic Algorithms (GA) is developed.

A brief overview of the GA's theory could be the following: Suppose that we have to maximize (minimize) $f(x)$, GA's are search algorithms which initially were insiped by the process of natural genetics (reproduction of an original population, performance of crossover and mutation, selection of the best). The main idea for an optimization problem is to start our search no with one initial point, but with a population of initial points. The 2*n* numbers (points) of this initial set (called population, quite analogously to the biological system) are converted to the binary system. In the sequel, they are considered as chromosomes (actually sequences of 0 and 1).

The next step is to form pairs of these points who will be considered as parents for a "reproduction" (Fig.1)

$$\left.\begin{matrix} 01100\,|\,100...11 \\ 00011\,|\,101...10 \end{matrix}\right\}\rightarrow \begin{matrix} 01100101...10 \\ 00011100..11 \end{matrix}$$

parents children

Fig.1. Crossover

"Parents" come to "reproduction" where they interchange parts of their "genetic material". (This is achieved by the so-called *crossover*, Fig.1) whereas always a very small probability for a Mutation exists. (Mutation is the phenomenon where quite randomly - with a very small probability though - a 0 becomes 1 or a 1 becomes 0). Assume that every pair of "parents" gives *k* children.

By the reproduction the population of the "parents" are enhanced by the "children" and we have an increasement of the original population because new members were added (parents always belong to the considered population). The new population has now 2*n*+*kn* members. Then the process of natural selection is applied. According the concept of natural selection, from the 2*n*+*kn* members, only 2*n* survive. These 2*n* members are selected as the members with the higher values of *f*, if we attempt to achieve *maximization* of *f* (or with the lower values of *f*, if we attempt to achieve *minimization* of *f*). By repeated iterations of reproduction (under crossover and mutation) and natural selection we can find the minimum (or maximum) of *f* as the point to which the best values of our population converge. The termination criterion is fulfilled if the mean value of *f* in the 2*n*-members population is no longer improved (maximized or minimized). More detailed overviews of GA can be found in [16] and [17]. Other recent results and applications can be found in [18] and [19].

In our problem of *m*-D systems stability we wish to minimize *f* over $w_1, w_2, ..., w_m$ when $w_i \in [0, 2\pi]$, *i*=1,...,m. To this end $w_1, w_2, ..., w_m$ are converted to the binary system and are considered as parts of a big chromosome (Fig 2).

$$\begin{matrix} w_1 & w_2 & & w_m \\ 100110010| & 001000111| & ...| & 111001010 \end{matrix}$$

Fig.2

If we suppose that for every w_i is converted to a *t*-bits binary number, for the "chromosome" of $w_1, w_2, ..., w_m$, we need *mt* bits. Our search starts with a randomly generated population of such 2*n* chromosomes.

In a quite random manner, this population is split into pairs of parents that will be crossed i.e. they will interchange their genetic material (with *c* crossovers) always under a very small probability *p* for mutation (for example p=0.01).

By this reproduction, a new population of 2*n*+*kn* members will be formed, since each pair of parents give birth to *k* children. The new population is filtered and only the 2*n* better members (here "better" means the 2*n* lower values of $f(w_1, w_2, ..., w_n)$) remain to the population, the other are deleted. By repeated iterations of reproduction (under crossover and mutation) and natural selection we can find the minimum of $f(w_1, w_2, ..., w_n)$, $0 \le w_i \le 2\pi$, *i*=1,...,*m* as the point to which the best values of our population converge. The termination criterion is: "the mean value of *f* in the population is no longer improved". The algorithm is summarized as follows

STEP A: Find (randomly) the initial population of 2n members

STEP B: Split the population (randomly) into n pairs

STEP C: Make c crossovers and from each pair of parents take k children. Every bit of every child has p probability for a mutation

STEP D: Find the new population 2n+2k (parents+children)

STEP E: From the new population select the 2n members with the lower values of f.

STEP F: If the absolute value of the difference of the mean value of f in the population of this generation with the mean value of f in the population of the previous generation is < å, then STOP, otherwise go to STEP C.

Example 1. Suppose that our 3-D singularity of the second kind), has the following characteristic polynomial

$$B(z_1,z_2,z_3)=0.8z_1+1.5z_1^2z_2+1.8z_2^3+0.2z_3+1.3z_2z_3^2+5.6$$

The first three conditions (i.e. (1.1) ÷ (1.3)) of DeCarlo-Strintzis Theorem are satisfied for $|z_i|\le 1$ with i=1,2,3 respectively, since

$$B(z_1,1,1)=1.5z_1^2+0.8z_1+8.9\neq 0,\quad B(1,z_2,1)=1.8z_2^3+2.8z_2+6.6\neq 0,$$

$$B(1,1,z_3)=1.3z_3^2+0.2z_3+9.7\neq 0$$

So, we have to examine the last equation in the DeCarlo-Strintzis Theorem. To this end, let us consider

$$f=f(w_1,w_2,w_3)=\left|B\left(e^{jw_1},e^{jw_2},e^{jw_3}\right)\right|.$$

We easily find that

$$f=\sqrt{Q_1^2+Q_2^2}$$

where

$$Q_1=0.8\cos(w_1)+1.5\cos(2w_1+w_2)+1.8\cos(3w_2)+0.2\cos(w_3)+1.3\cos(w_2+2w_3)+5.$$

$$Q_2=0.8\sin(w_1)+1.5\sin(2w_1+w_2)+1.8\sin(3w_2)+0.2\sin(w_3)+1.3\sin(w_2+2$$

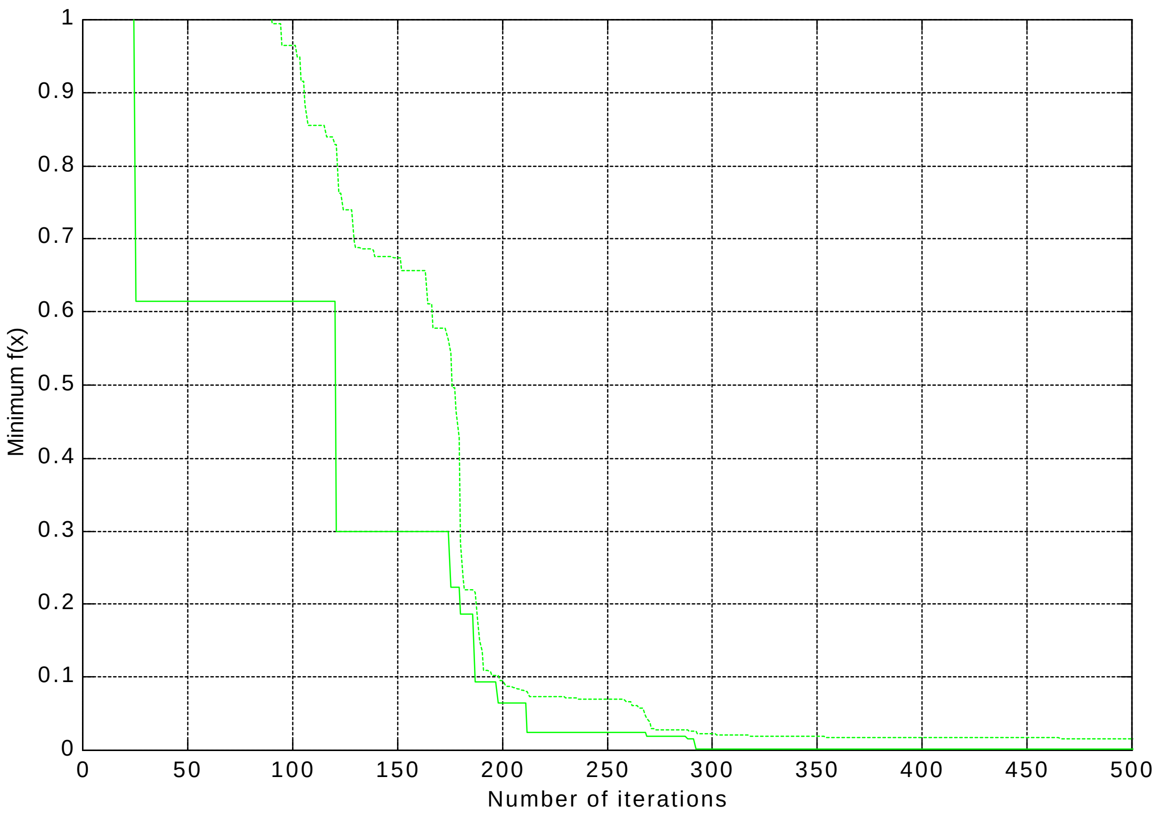

Fig.3. Convergence of the optimum value of *f*: ____ in every generation, as well as of the mean value of *f* in every generation _ _ _ _

Iterations	1	50	100	150	200	250	300	350	400	450	500
Parents	6.134	1.216	0.966	0.674	0.094	0.070	0.023	0.019	0.018	0.017	0.016
Best	2.870	0.614	0.614	0.299	0.065	0.025	0.002	0.002	0.002	0.001	0.001

Using now the previously presented GA with n=5, k=4, t=12, p= 0.01, c =6 we obtain that the Optimum value of f in each generation (denoted by ____) as well as the mean value of f in each generation parents (denoted by _ _ _ _) converges to 0 (Fig.3 and Table 1). Therefore for this example M=0 and the polynomial is (Schur) unstable.

Example 2. Suppose that our 3-D system (without any nonessential singularity of the second kind), has the following characteristic polynomial

$$B(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3 - z_1 z_2 z_3 + 5$$

The first three conditions (i.e. (1.1) ÷ (1.3)) of DeCarlo-Strintzis Theorem are satisfied for $|z_i| \leq 1$ where i=1,2,3 respectively, since

$B(z_1,1,1) = z_1^2 - z_1 + 7 \neq 0$, $B(1,z_2,1) = z_2^2 - z_2 + 7 \neq 0$, $B(1,1,z_3) = 7 \neq 0$

So, we have to examine the last equation in the DeCarlo-Strintzis Theorem. Similarly one has $f = \sqrt{Q_1^2 + Q_2^2}$ where

$$Q_1 = \cos(2w_1) + \cos(2w_2) + \cos(w_3) - \cos(w_1 + w_2 + w_3) + 5$$
$$Q_2 = \sin(2w_1) + \sin(2w_2) + \sin(w_3) - \sin(w_1 + w_2 + w_3)$$

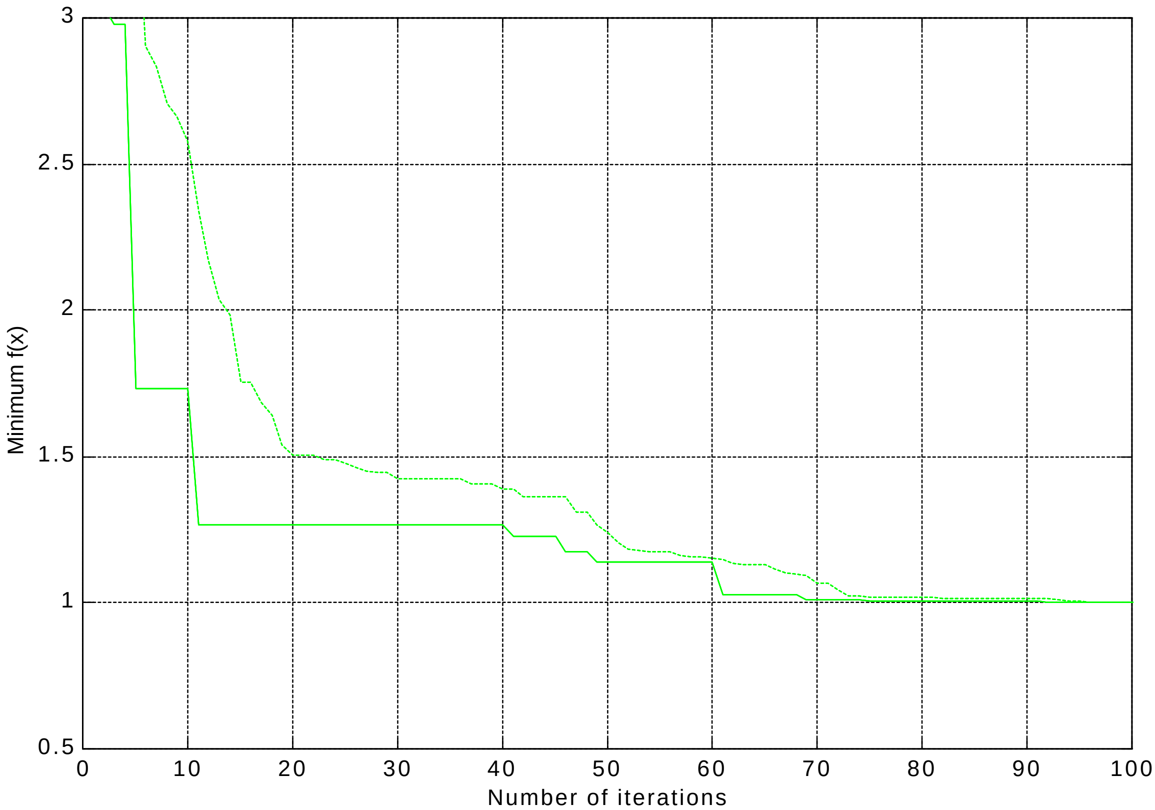

Fig.4. Convergence of the optimum value of f: ____ in every generation, as well as of the mean value of f in every generation _ _ _ _

Iterations	1	10	20	30	40	50	60	70	80	90	100
Parents	5.275	2.579	1.504	1.424	1.388	1.240	1.152	1.069	1.019	1.010	1.000
Best	3.039	1.731	1.265	1.265	1.265	1.137	1.137	1.009	1.008	1.008	1.000

Using now the previously presented GA with n=5, k=4, t=12, p= 0.01, c =6 we obtain that the Optimum value of f in each generation (denoted by ____) as well as the mean value of f in each generation parents (denoted by _ _ _ _) converges to 1 (Fig.4 and Table 2). Therefore for this example M>0 and the polynomial is (Schur) Stable.

Example 3. Let our 5-D system (without any nonessential singularity of the second kind), has the following characteristic polynomial

$$B(z_1,z_2,z_3,z_4,z_5)=z_1^2z_3^3+z_3^3z_4^2+z_1^3z_2z_5+z_1z_2z_3z_4z_5+5$$

The first five conditions (i.e. (1.1) ÷ (1.5)) of DeCarlo-Strintzis Theorem are satisfied for $|z_i|\le 1$ where i=1,2,3,4,5 respectively, because

$$B(z_1,1,1,1,1)=z_1^3+z_1^2+z_1+7\neq 0$$
$$B(1,z_2,1,1,1)=2z_2+8\neq 0$$
$$B(1,1,z_3,11)=z_3+2z_3^3+7\neq 0$$
$$B(1,1,1,z_4,1)=z_4^2+z_4+8\neq 0$$
$$B(1,1,1,1,z_5)=z_5^3+2z_5+7\neq 0$$

So, we have to examine the last equation in the DeCarlo-Strintzis Theorem. Here, $f=\sqrt{Q_1^2+Q_2^2}$, where

$$Q_1=\cos(2w_1+3w_3)+\cos(3w_3+2w_4)+\cos(3w_3+w_2+w_5)+\cos(w_1+w_2+w_3+w_4+w_5)+5$$

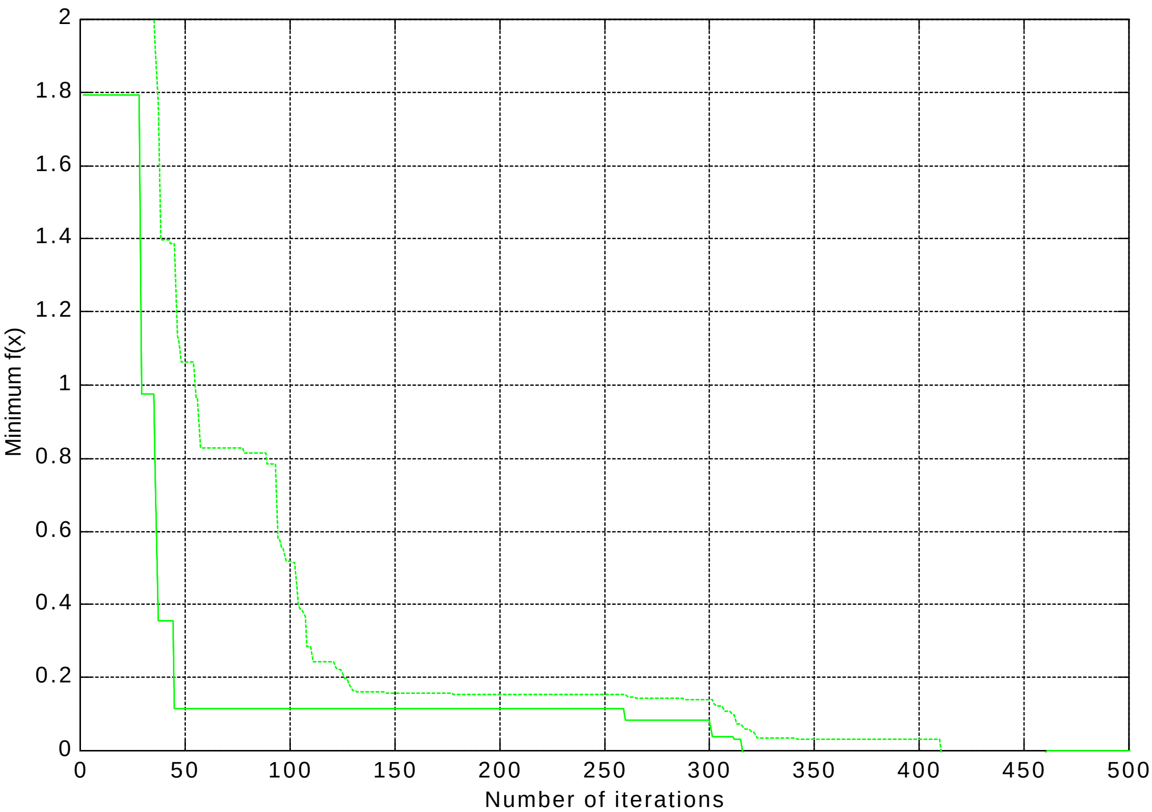

Fig.5. Convergence of the optimum value of f: ____ in every generation, as well as of the mean value of f in every generation _ _ _ _

Iterations	1	50	100	150	200	250	300	350	400	450	500
Parents	5.422	1.062	0.518	0.159	0.156	0.156	0.142	0.033	0.032	0.013	0.009
Best	1.793	0.116	0.116	0.116	0.116	0.116	0.086	0.021	0.021	0.011	0.000

Using now the previously presented GA with n=5, k=4, t=12, p= 0.01, c=6 we obtain that the Optimum value of f in each generation (denoted by ____) as well as the mean value of f in each generation parents (denoted by _ _ _ _) converges to 0 (Fig.5 and Table 3). Therefore for this example $\underline{M}$=0 and the polynomial is (Schur) unstable.

III. Conclusion

Genetic Algorithms provide us an elegant, reliable and efficient method for checking the stability of m-D ($m \geq 3$) systems. First the m-D stability problem is reduced to an appropriate minimization problem by using the last condition of the DeCarlo-Strintzis Theorem. Investigation of concepts like m-D stability margin are left for future research. Furthermore the present method can be improved if we use some heuristics techniques like *zooming*, etc [20].

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