Positive Singular Value Decomposition for Two-Dimensional Arrays

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Abstract

In some cases, the decomposition of a positive elements' matrix in a way similar to the formal SVD (*Singular Value Decomposition*) with positive elements in all vectors is desirable. This Positive Singular Value Decomposition is the objective of this work. In this paper, the Positive SVD is examined for $2-D$ arrays.

Key words: Positive multidimensional systems, singular value decomposition, two-dimensional arrays techniques, positive singular value decomposition, measurements.

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1 Introduction and Problem Formulation

It is known that in the context of modern computer science, the method of the singular value decomposition (SVD) plays an important role. This very popular technique permits the data compression of a $2-D$ or $m-D$ with $m>2$ array. As a result, a great economy in data storage as well as in the data transmission is obtained. SVD is analytically presented in standard textbooks on Linear Algebra and Image Processing [$1 \div 6$]. The SVD of an N_1-by-N_2 matrix A is

$$A=\sigma_1 u_1 v_1^T+\ldots+\sigma_r u_r v_r^T \qquad (1)$$

where for the so called singular values $\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_r$ it has been assumed that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$ where $\sigma_i=\sqrt{\lambda_i},\ \ \lambda_i$ are the eigenvalues of A^TA ($i=1,\ldots,r$) where T denotes the transpose matrix). The vectors $u_1, v_1, \ldots, u_r, v_r$ are certain orthonormal vectors and $r=\text{rank }(A) \leq \min(N_1,N_2)$. A more detailed description of the method can be found in [1,2]. In a tensor notation [3], Equation (1) is written as

$$A=\sigma_1 u_1 \otimes v_1+\ldots+\sigma_r u_r \otimes v_r \qquad (2)$$

where $\otimes$ is the usual tensor product. In many professional software packages (such

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as MatLab or Mathematica) the method is automated via appropriate commands.

One should also note that one of the main application of the SVD is the data compression of a $2-D$ array and the resulting economy in data storage when: $\sigma_1 \geq \ldots \geq \sigma_q \gg \sigma_{q+1} \geq \ldots \geq \sigma_r$ (where $q < r$), because in this case Equation (2) can be approximately written as

$$A = \sigma_1 u_1 \otimes v_1 + \ldots + \sigma_r u_r \otimes v_r \qquad (3)$$

So, instead of $N_1 \cdot N_2$ memory positions for the matrix A, we need only $q \cdot (N_1 - 1 + N_2 - 1 + 1) = q \cdot (N_1 + N_2 - 1)$ since every orthonormal $n-$vector requires $n-1$ memory positions.

It is also well known that if we demand the optimum approximation (in the sense of the least squares approach) of a matrix A by a product σuv^T i.e. $minimization\left|\left|A - \sigma uv^T\right|\right|$ this minimization yields $\sigma = \sigma_1,\ u = u_1,\ v = v_1$ where $\sigma_1,\ u_1,\ v_1$ are those that result from the SVD. This very important property of the SVD permit us to solve several optimization problems via SVD, $[1 \div 6]$. Furthermore, in k step $(1 \leq k \leq r)$, σ_k, u_k, v_k are the same with those resulting from the solution of the minimization problem (with respect to σ, u, v)

$$\min\left|\left|A_{k-1} - \sigma uv^T\right|\right| \qquad (4)$$

where
$A_{k-1} = A_{k-2} - \sigma_{k-1} u_{k-1} v_{k-1}^T \quad (A_0 = A)$
This is the so-called *least squares property* of the SVD.

In this paper, an attempt is made to find the decomposition of a positive elements' matrix in a way similar to the formal SVD (Singular Value Decomposition) with the above stated least squares property, where all vectors of the SVD ought to contain positive elements. In other words, our problem can be formulated as follows: Find a decomposition of the matrix A as in Equation (1) with the least square property under the constraint all σ's and all the coordinates of u's and v's to be positive.

This is a problem arising in Statistics as well as in Measurements Engineering when our Data are all positive and when we attempt an approximation under the constraint of the positiveness of all quantities (entries) in the vectors $u_1, v_1, \ldots, u_r, v_r$.

Moreover, Positive 2-dimensional Systems Theory seems to gain ground nowadays, promising new interesting applications in biomedical signal processing as well as in physics and astronomy, computers architecture etc, $[12 \div 15]$.

2 Problem Solution

Let A, B be two matrices (i.e. two-dimensional arrays) of the same dimensions, we denote as A/B the matrix for which the i,j element results from the division of the i,j element of A by the i,j element of B.

Now, the solution of the PSVD (Positive Singular Value Decomposition) problem should actually can follow the following strategic:
Find $\sigma = \sigma_1,\ u = u_1,\ v = v_1$ such that $\left|\left|(A - \sigma uv^T)/A\right|\right|$ to be minimum with σ and all the coordinates of u and v to be

positive.

Afterwards, *consider* the new matrix $A_1 = A - \sigma_1 u_1 v_1^T$ and *find* $\sigma = \sigma_2$, $u = u_2$, $v = v_2$ such that $\left|\left|(A_1 - \sigma u v^T)/A\right|\right|$ to be minimum with σ and all the coordinates of u and v to be positive. Finally (after r iteration steps) *find* the decomposition

$$A = \sigma_1 u_1 v_1^T + \ldots + \sigma_r u_r v_r^T \qquad (5)$$

where all σ's and all the coordinates of u's and v's should be positive. Note that:
r =rank(A) $\leq$min(N_1, N_2). This decomposition results as follows: In k step $(k, 1 \leq k \leq r)$, σ_k, u_k, v_k result from the solution of the following minimization problem (with respect to σ, u, v)

$$\min \left|\left|(A_{k-1} - \sigma u v^T)/A_{k-1}\right|\right| \qquad (6)$$

where
$A_{k-1} = A_{k-2} - \sigma_{k-1} u_{k-1} v_{k-1}^T \quad (A_0 = A)$

The solution of the problem proceeds as follows: First we find $\sigma = \sigma_1$, $u = u_1$, $v = v_1$ from the minimization problem $\left|\left|(A - \sigma u v^T)/A\right|\right|$ where σ and all the coordinates of u and v to be positive. To this end, let us denote the i_1, i_2–element of the matrix A as $a(i_1, i_2)$, the i_1–element of the vector u as $b_1(i_1)$ while the i_2–element of the vector v as $b_2(i_2)$. The constant σ is incorporated at the moment in the product of u and v.

So,
we have to minimize the N_1 x N_2 quantities: $(a(i_1, i_2) - b_1(i_1)b_2(i_2))/a(i_1, i_2)$ or equivalently $1 - b_1(i_1)b_2(i_2)/a(i_1, i_2)$.

Therefore, we have to solve (least squares) the problem

$$b_1(i_1)b_2(i_2)/a(i_1, i_2) \cong 1 \qquad (7)$$

or equivalently

$$\ln a(i_1, i_2) \cong \ln b_1(i_1) + \ln b_2(i_2) \qquad (8)$$

or

$$a'(i_1, i_2) \cong b_1'(i_1) + b_2'(i_2) \qquad (9)$$

where

$$a'(i_1, i_2) = \ln a(i_1, i_2) \qquad (10)$$

$$b_1'(i_1) = \ln b_1(i_1) \qquad (11)$$

and

$$b_2'(i_2) = \ln b_2(i_2) \qquad (12)$$

and $i_1 \in \{1, \ldots, N_1\}$ and $i_2 \in \{1, \ldots, N_2\}$. We write Equation (9) as follows:

$$b_1'(i_1) + b_2'(i_2) \cong a'(i_1, i_2) \qquad (13)$$

with $i_1 \in \{1, \ldots, N_1\}$ and $i_2 \in \{1, \ldots, N_2\}$. These N_1 x N_2 Equations constitute a typical optimization problem linear in $a'(i_1, i_2)$, $b_1'(i_1)$ and $b_2'(i_2)$ Furthermore, if we use the notation

$$C=\begin{bmatrix} \overbrace{\begin{matrix}1 & 0 & \ldots & 0\end{matrix}}^{N_1} & \overbrace{\begin{matrix}1 & 0 & \ldots & 0\end{matrix}}^{N_2} \\ \begin{matrix}1 & 0 & \ldots & 0\end{matrix} & \begin{matrix}0 & 1 & \ldots & 0\end{matrix} \\ \begin{matrix}\vdots & \vdots & \ldots & \vdots\end{matrix} & \begin{matrix}\vdots & \vdots & \ldots & \vdots\end{matrix} \\ \begin{matrix}1 & 0 & \ldots & 0\end{matrix} & \begin{matrix}0 & 0 & \ldots & 1\end{matrix} \\ \begin{matrix}0 & 1 & \ldots & 0\end{matrix} & \begin{matrix}1 & 0 & \ldots & 0\end{matrix} \\ \begin{matrix}0 & 1 & \ldots & 0\end{matrix} & \begin{matrix}0 & 1 & \ldots & 0\end{matrix} \\ \begin{matrix}\vdots & \vdots & \ldots & \vdots\end{matrix} & \begin{matrix}\vdots & \vdots & \ldots & \vdots\end{matrix} \\ \begin{matrix}0 & 1 & \ldots & 0\end{matrix} & \begin{matrix}0 & 0 & \ldots & 1\end{matrix} \\ \begin{matrix}\vdots & \vdots & \ldots & \vdots\end{matrix} & \begin{matrix}\vdots & \vdots & \ldots & \vdots\end{matrix} \\ \begin{matrix}0 & 0 & \ldots & 1\end{matrix} & \begin{matrix}1 & 0 & \ldots & 0\end{matrix} \\ \begin{matrix}0 & 0 & \ldots & 1\end{matrix} & \begin{matrix}0 & 1 & \ldots & 0\end{matrix} \\ \begin{matrix}\vdots & \vdots & \ldots & \vdots\end{matrix} & \begin{matrix}\vdots & \vdots & \ldots & \vdots\end{matrix} \\ \begin{matrix}0 & 0 & \ldots & 1\end{matrix} & \begin{matrix}0 & 0 & \ldots & 1\end{matrix} \end{bmatrix} \tag{14}$$

as well as

$$b'=\begin{bmatrix} b_1'(1) \\ \vdots \\ b_1'(N_1) \\ b_2'(1) \\ \vdots \\ b_2'(N_2) \end{bmatrix} \tag{15}$$

and

$$a'=\begin{bmatrix} a'(1,1) \\ a'(1,2) \\ \vdots \\ a'(1,N_2) \\ a'(2,1) \\ a'(2,2) \\ \vdots \\ a'(2,N_2) \\ \vdots \\ a'(N_1,1) \\ a'(N_1,2) \\ \vdots, \\ a'(N_1,N_2) \end{bmatrix} \tag{16}$$

then the above optimization problem re–written as follows

$$C \cdot b' \cong a' \tag{17}$$

It is known ([1 ÷ 4]) that the approximate solution of this problem is given as follows

$$b' = (C^T C)^{-1} C^T a' \tag{18}$$

where

$$C^T C=\begin{bmatrix} N_2 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \\ 0 & N_2 & \ldots & 0 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & N_2 & 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & N_1 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 1 & 0 & N_1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & N_1 \end{bmatrix} \tag{19}$$

or in a simpler notation:

$$C^T C = \begin{bmatrix} N_2 \cdot \mathcal{I} & \mathcal{K} \\ \mathcal{K} & N_1 \cdot \mathcal{I} \end{bmatrix} \tag{20}$$

$\mathcal{I}$ is the Identity matrix and $\mathcal{K}$ is defined as the matrix with all of its elements equal to 1 i.e.

$$\mathcal{K} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} \tag{21}$$

Here $\mathcal{K}$ is a $N_2 - by - N_2$ matrix. Unfortunately, the matrix $(C^T C)$ is singular. That means that we have more than one optimal solutions. This also means that for all these "optimal solutions" the total square error of approximation is exactly the same. For this reason, we can demand, in addition, the minimization of $|b'|$. So the considered problem is reduced to

$$\min b'^2 \tag{22}$$

under the constraint of (17). By introducing the well-known concept of Lagrange multipliers, the constrained minimization problem is reduced to an uncostrained one.

$$\min b'^2 + \lambda \cdot C^T (C b' - a') \tag{23}$$

The final minimization over b' and λ is achieved by using a variety of numerical techniques. Elegant results can be obtained by the Levenberg-Marquardt routine.

Another approach would be the parameter solution of (17) with respect to b' depending on the real parameters $\mu_1, \mu_2, \dots, \mu_\rho$ where $\rho = N_1 + N_2 - \text{rank}(C^T C)$. In such a case, we introduce these parameters to (22) and proceed with an uncostrained minimization problem.

So, $b'_1(1), \dots, b'_{N_1}, \dots, b'_2(N_2)$ are known. Therefore, $b_1(1), \dots, b_{N_1}, \dots, b_2(N_2)$ are also known (Equations (11) and (12)). So, the vectors u and v are known. We normalize them defining:

$$\hat{u} = \frac{u}{||u||} \tag{24}$$

$$\hat{v} = \frac{v}{||v||} \tag{25}$$

Therefore

$$\sigma = ||u|| \, ||v|| \tag{26}$$

The normalized vectors $u_1 = \hat{u}$, $v_1 = \hat{v}$, are now used as u and v (re–definition of u and v). Obviously, they are with positive elements and they are such that the norm $\left|\left|(A - \sigma u v^T)/A\right|\right|$ to be minimum. The iterations of the "positive Singular Value Decomposition" of A consists exactly by the same steps and they are dealt similarly. So, one must consider the matrix $A_1 = A - \sigma_1 u_1 v_1^T$ and find in the same way $\sigma = \sigma_2$, $u = u_2$, $v = v_2$ and after r steps the decomposition shown by Equation (5) where all σ's and all the coordinates of u's and v's will be positive.

3 Conclusion

In this paper, the concept of the PSVD (Positive Singular Value Decomposition) is introduced and investigated. *MORE DETAILS SEE IN* [17]. By the term PSVD, it is meant the decomposition of a positive elements' matrix in a way similar to the formal SVD but with positive elements in all vectors. The PSVD could seem useful for Positive $m-D$ Systems design and applications $[12 \div 16]$.

References

[1] G.Strang, *Linear Algebra and its Applications*, Hancourt Brace Jovanovich, San Diego, 1988.

[2] J.Ortega, *Matrix Theory*, Plenum Press, London, 1989.

[3] R.M.Bowen and C.-C. Wang, *Introduction to Vectors and Tensors, Linear and Multilinear Algebra*, Vol.I & II, Plenum Press, New York and London, 1976.

[4] F.Chatelin, *Eigenvalues of Matrices*, J.W.& Sons, N.Y,1990

[5] A.K.Jain, *Fundamentals of Digital Image Processing*, Prentice Hall, Englewood Cliffs, New Jersey, 1989.

[6] W.Pratt, *Digital Image Processing*, J.W.& Sons, N.Y, 1978.

[7] K.Galkowski, "Linear Transformations of Transfer Function Variables of an $m-D$ System", *Int.J. of Circuit Theory and Applications*, Vol. 21, pp.351-360, 1993.

[8] N.E.Mastorakis, "Factorization of $m-D$ polynomials in linear $m-D$ factors", *Int.J.Syst.Sci.*, Vol. 23, No.11, 1805–1824, 1992.

[9] N.E.Mastorakis, "A General Factorization method for multivariable polynomials", *Multidimensional Systems & Signal Processing*, Vol.5, No.2, 151–178, 1994.

[10] N.E.Mastorakis, "Comments Concerning Multidimensional Polynomial Properties", *IEEE Trans. on Aut. Control*, Vol.41, No.2, 260, Feb. 1996.

[11] N.E.Mastorakis, "Singular Value Decomposition in Multidimensional Arrays", *Int. J. of Syst. Sci.*, Vol.27, No.5, 647–650, 1996.

[12] N.E.Mastorakis, *Recent Advances in Information Science and Technology*, World Scientific Pub.Co., 1998.

[13] G.Antoniou, A.Papis and A.Chilmaza, "Two-Dimensional Positive Systems: Minimal State Space Realization and Transfer Functions", pp.282-286, in N.E.Mastorakis, *Recent Advances in Information Science and Technology*, World Scientific Pub.Co., 1998.

[14] E.Fornasini, "Primitive of Two-Dimensional Positive Systems",

pp.287-292, in N.E.Mastorakis, *Recent Advances in Information Science and Technology*, World Scientific Pub.Co., 1998.

[15] T.Kaczorek, "Weakly Positive Continuous-Time Linear Systems" *EURISCON 98, Third European Robotics, Intelligent Systems and Control Conference*, June 22-25, 1998, Athens, Greece.

[16] All the publications at the WEB Site:
http://www.softlab.ntua.gr
/˜mastor/Publications.htm

[17] N.E.Mastorakis, "Positive Singular Value Decomposition", *Modern Applied Mathematics Techniques in Circuits, Systems and Control*, WSES Press, 1999.