A Non-Linear Two-Dimensional Model

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Abstract. In many applications, non-linear models are actually obtained. In this paper, a non-linear 2-D (two-dimensional) model is presented and its study is attempted via numerical methods. The proposed non-linear model corresponds to a linear 2-D one recently proposed by the author [8,9]. The model consists of a system of non-linear Partial Differential Equations (PDE's).
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1 Introduction

2-D systems' analysis and synthesis have attained great sophistication maturity. Several books have been also edited which give excellent surveys of the recent results of 2-D systems among them they are [2], [3], [4]. Various mathematical fields, such as factorization of multivariable polynomials and multivariable matrices, 2-D system stability, singular 2-D equations etc. are also motivated by 2-D system theory, [10]÷[16].

The most famous and almost exclusively utilised state-space model for these systems is the Roesser model [1]. The Roesser model is stated for the discrete Linear Shift Invariant (LSI) 2-D systems as follows

$$X_1(n_1+1,n_2) = A_1 \cdot X_1(n_1,n_2) + A_2 \cdot X_2(n_1,n_2) + B_1 \cdot u(n_1,n_2) \qquad (1a)$$

$$X_2(n_1,n_2+1) = A_3 \cdot X_1(n_1,n_2) + A_4 \cdot X_2(n_1,n_2) + B_2 \cdot u(n_1,n_2) \qquad (1b)$$

$$y(n_1,n_2) = C_1 \cdot X_1(n_1,n_2) + C_2 \cdot X_2(n_1,n_2) + D \cdot u(n_1,n_2) \qquad (2)$$

with the initial conditions: $X_1(0,n_2)$, $X_2(n_1,0)$, $n_1,\, n_2 \in \mathrm{N}$. $X = \begin{bmatrix} X_1(n_1,n_2) \\ X_2(n_1,n_2) \end{bmatrix}$ is the state-space vector, X_1, X_2 are vectors of N_1, N_2 dimensions, *u, y* are the scalar input and output respectively and $A_1, A_2, A_3, A_4, B_1, B_2, C_1, C_2, D$ are matrices of appropriate dimensions.

The Roesser model is first stated as a practical model for image processing. However, Roesser gave a theoretical foundation to it. However, almost all the papers refer to the discrete 2-D LSI systems. Only in few works the corresponding continuous model is simply mentioned [5], [6], [7]. The continuous model which correspond to the Roesser discrete model is the following system of PDE's. ([5], [6], [7]).

$$\frac{\partial X_1(x,y)}{\partial x} = A_1 \cdot X_1(x,y) + A_2 \cdot X_2(x,y) + B_1 \cdot u(x,y) \qquad (3a)$$

$$\frac{\partial X_2(x,y)}{\partial y} = A_3 \cdot X_1(x,y) + A_4 \cdot X_2(x,y) + B_2 \cdot u(x,y) \qquad (3b)$$

$$v(x,y) = C_1 \cdot X_1(x,y) + C_2 \cdot X_2(x,y) + D \cdot u(x,y) \qquad (4)$$

with the initial conditions: $X_1(0,y)=f(y)$, $X_2(x,0)=g(x)$, $x,\, y \in \mathrm{R}_+$ (i.e. $x \geq 0,\, y \geq 0$). $X = \begin{bmatrix} X_1(x,y) \\ X_2(x,y) \end{bmatrix}$ is the state-space vector, $X_1(x,y), X_2(x,y)$ are vectors of N_1, N_2 dimensions, *u*, *v* are the scalar input and output respectively and $A_1, A_2, A_3, A_4, B_1, B_2, C_1, C_2, D$ are matrices of appropriate dimensions.

The main difficulty in this model is : Suppose that $u(x,y)=\delta(x)\delta(y)$, where $\delta(\ )$ is a delta (Dirac) function, then substituting for example in (3a), one finds $X_1(x,y)$ to be incontinuous with respect to *x* as well as to be delta function with respect to *y*. This is not true since, especially in image processing, both $X_1(x,y), X_2(x,y)$ are assumed to be bounded functions. So, various difficulties can be obtained if one attempts to find the general response formula in this model. It seems unavoidable that a modification in this continuous model should be made.

In order to overcome the above difficulty obtained at the continuous analogous of the 2-D Roesser model, the following continuous model has been proposed in [8] and [9].

$$\frac{\partial X_1(x,y)}{\partial x} = A_1 \cdot X_1(x,y) + A_2 \cdot X_2(x,y) + B_1 \cdot \int_0^y u(x,y_1)dy_1 \tag{5a}$$

$$\frac{\partial X_2(x,y)}{\partial y} = A_3 \cdot X_1(x,y) + A_4 \cdot X_2(x,y) + B_2 \cdot \int_0^x u(x_1,y)dx_1 \tag{5b}$$

$$v(x,y) = C_1 \cdot X_1(x,y) + C_2 \cdot X_2(x,y) + D \cdot u(x,y) \tag{6}$$

with the initial conditions: $X_1(0,y)=f(y)$, $X_2(x,0)=g(x)$, $x, y \in \mathrm{R}_+$ (i.e. $x \geq 0,\ y \geq 0$). $X = \begin{bmatrix} X_1(x,y) \\ X_2(x,y) \end{bmatrix}$ is the state-space vector, $X_1(x,y), X_2(x,y)$ are vectors of N_1, N_2 dimensions, u, v are the scalar input and output respectively and $A_1, A_2, A_3, A_4, B_1, B_2, C_1, C_2, D$ are matrices of appropriate dimensions.

This model has been analytically studied in [8].

In the present paper, the following general non-linear model is introduced

$$\frac{\partial X_1(x,y)}{\partial x} = f_1\left(X_1(x,y), X_2(x,y), \int_0^y u(x,y_1)dy_1 \right) \tag{7a}$$

$$\frac{\partial X_2(x,y)}{\partial y} = f_2\left(X_1(x,y), X_2(x,y), \int_0^x u(x_1,y)dx_1 \right) \tag{7b}$$

$$v(x,y) = f_3\left(X_1(x,y), X_2(x,y), u(x,y)\right) \tag{8}$$

with the initial conditions: $X_1(0,y)=f(y)$, $X_2(x,0)=g(x)$, $x, y \in \mathrm{R}_+$ (i.e. $x \geq 0,\ y \geq 0$). $X = \begin{bmatrix} X_1(x,y) \\ X_2(x,y) \end{bmatrix}$ is the state-space vector, $X_1(x,y), X_2(x,y)$ are vectors of N_1, N_2 dimensions, u, v are the scalar input and output respectively.

The objective of the present paper is the study of the above non-linear 2-D model i.e. the system of (7a), (7b) and (8).

2 Main Results

First, one ought to find $X_1(x,0)$ and $X_2(0,y)$. This can be achieved as follows.

$$\frac{\partial X_1(x,0)}{\partial x} = f_1\left(X_1(x,0), X_2(x,0), 0\right) \tag{9}$$

or

$$\frac{\partial X_1(x,0)}{\partial x} = f_1\left(X_1(x,0), g(x), 0\right) \tag{10}$$

with the initial condition $X_1(0,0) = f(0)$

This is a (non-linear) ordinary matrix differential equation and it can be solved using any numerical method (Taylor, Runge-Kutta, etc.). Similarly, we have

$$\frac{\partial X_2(0,y)}{\partial y} = f_2\left(f(y), X_2(0,y), 0\right) \tag{11}$$

with the initial condition $X_2(0,0) = g(0)$

This is also a (non-linear) ordinary matrix differential equation and it can be solved using numerical methods (Taylor, Runge-Kutta, etc.).

Afterwards, an attempt is made to find $X_1(x,y)$ and $X_1(x,y)$. To this end, a discretization of our space is made. So, our plane is divided into the points $(n_1 \cdot \Delta x, n_2 \cdot \Delta y)$ where n_1, $n_2 \in \mathrm{N}$. So, if one uses the Taylor's theorem, then one has

$$X_1((n_1+1)\Delta x, n_2 \cdot \Delta y) = X_1(n_1 \cdot \Delta x, n_2 \cdot \Delta y) + \left.\frac{\partial X_1}{\partial x}\right|_{(n_1 \cdot \Delta x, n_2 \cdot \Delta y)} \cdot \Delta x \tag{12}$$

$$X_1((n_1+1)\Delta x, n_2 \cdot \Delta y) = X_1(n_1 \cdot \Delta x, n_2 \cdot \Delta y) + \\ + f_1\left(X_1(n_1 \cdot \Delta x, n_2 \cdot \Delta y), X_2(n_1 \cdot \Delta x, n_2 \cdot \Delta y), \int_0^{n_2 \cdot \Delta y} u(n_1 \cdot \Delta x, y_1)dy_1 \right) \cdot \Delta x \tag{13a}$$

Similarly one finds:

$$X_2(n_1 \cdot \Delta x,(n_2+1)\cdot \Delta y) = X_2(n_1 \cdot \Delta x, n_2 \cdot \Delta y) +$$
$$+ f_2\left(X_1(n_1 \cdot \Delta x, n_2 \cdot \Delta y), X_2(n_1 \cdot \Delta x, n_2 \cdot \Delta y), \int_0^{n_1 \cdot \Delta x} u(x_1, n_2 \cdot \Delta y) dx_1 \right) \cdot \Delta y \qquad (13b)$$

Equations (13a) and (13b) are also accompanied by the equation

$$v(n_1 \cdot \Delta x, n_2 \cdot \Delta y) = f_3(X_1(n_1 \cdot \Delta x, n_2 \cdot \Delta y), X_2(n_1 \cdot \Delta x, n_2 \cdot \Delta y), u(n_1 \cdot \Delta x, n_2 \cdot \Delta y)) \qquad (14)$$

This can be called as the Taylor method in two dimensions. Obviously, equations (13a), (13b) and (14) are proper for computer use. This is the proposed numerical solution for our non-linear model.

3 Example

Consider the following simple example:

$$\frac{\partial X_1(x,y)}{\partial x} = -X_1(x,y) + X_2^2(x,y) + \int_0^y u(x, y_1) dy_1 \qquad (15a)$$

$$\frac{\partial X_2(x,y)}{\partial y} = \sin(X_1(x,y) + X_2(x,y)) \qquad (15b)$$

$$v(x,y) = 2X_1(x,y) + 3X_2(x,y) - u^2(x,y) \qquad (16)$$

where $X_1(0,y)=y$, $X_2(x,0)=sin(x)$, $X_1(x,y), X_2(x,y)$ are scalars for simplicity (i.e. $N_1 = 1, N_2 = 1$) and $u(x,y) = xy$.

First, we find $X_1(x,0)$ and $X_2(0,y)$.

$$\frac{\partial X_1(x,0)}{\partial x} = \sin^2 x \qquad (17)$$

with the initial condition $X_1(0,0) = 0$. The solution of this simple equation is $X_1(x,0) = \frac{2x + \sin(2x)}{4}$. (Fig.1).

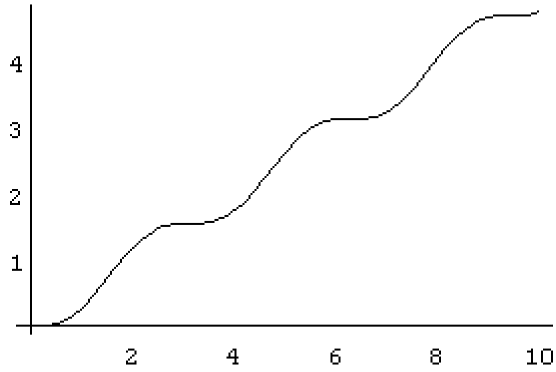

Fig.1: $X_1(x,0)$ versous *x*

Similarly, from (11) one obtains

$$\frac{\partial X_2(0,y)}{\partial y} = \sin(y + X_2(0,y)) \qquad (18)$$

with the initial condition $X_2(0,0) = 0$. This is also a non-linear ordinary differential equation and it can be solved using a typical Runge-Kutta method (Fig.2)

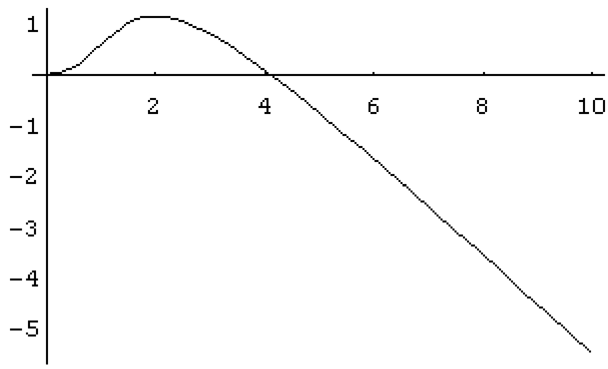

Fig.2. $X_2(0,y)$ versous *y*

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Following the afore mentioned procedure one has

$$X_1((n_1+1)\Delta x, n_2\cdot\Delta y) = X_1(n_1\cdot\Delta x, n_2\cdot\Delta y) + \\ + \left(-X_1(n_1\cdot\Delta x, n_2\cdot\Delta y) + X_2^2(n_1\cdot\Delta x, n_2\cdot\Delta y) + n_1\cdot\Delta x(n_2\cdot\Delta y)^2/2\right)\Delta x \tag{19a}$$

as well as

$$X_2(n_1\cdot\Delta x, (n_2+1)\cdot\Delta y) = X_2(n_1\cdot\Delta x, n_2\cdot\Delta y) + \\ + \sin(X_1(n_1\cdot\Delta x, n_2\cdot\Delta y) + X_2(n_1\cdot\Delta x, n_2\cdot\Delta y))\cdot\Delta y \tag{19b}$$

and the output-equation

$$v(n_1\cdot\Delta x, n_2\cdot\Delta y) = 2X_1(n_1\cdot\Delta x, n_2\cdot\Delta y) + \\ + 3X_2(n_1\cdot\Delta x, n_2\cdot\Delta y) - u^2(n_1\cdot\Delta x, n_2\cdot\Delta y) \tag{20}$$

So, it is obviously easy to find, via computer, X_1, X_2 and v at the points $n_1\cdot\Delta x, n_2\cdot\Delta y$.

4 Conclusion

In this paper, a non-linear 2-D model is presented. The solution of this model is achieved via the Taylor approximation and computer simulation. A non-trivial example illustrates the method. Instead of Taylor approximation, one can use Runge-Kutta formulas or other modern numerical methods. This very interesting methods are left for another publication.

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