On the Eigenstructure Assignment by Constant Output Feedback for Linear Time Invariant Systems via Appropriate Minimization

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Abstract: -In this paper, the problem of eigenstructure assignment by constant output feedback for linear time-invariant systems via appoprіate minimization is considered. After a theoretical analysis, the problem is reduced to the minimization of the distance of the desired characteristic matrix from the characteristic matrix of the under output feedback control law system at the unit circle. By the term characteristic matrix, we mean the matrix $\mathbf{zI}-\mathbf{A}$ for the system described by $\mathbf{x}(n+1)=\mathbf{Ax}(n)+\mathbf{Bu}$. The method can be easily extended in multimentional systems' case.

Key words: -Linear systems, constant output feedback, invariant polynomials.
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1 Introduction

Consider the linear time-invariant discrete-time, system described by the equations:

$$\mathbf{x}(n+\mathbf{1})=\mathbf{Ax}(n)+\mathbf{Bu}(n) \quad (1.1)$$

$$\mathbf{y}(n)=\mathbf{Cx}(n)+\mathbf{Du}(n) \quad (1.2)$$

where $\mathbf{A}$ is an $\nu x \nu$, $\mathbf{B}$ an $\nu x \nu_1$, $\mathbf{C}$ an $\nu_2 x \nu$ matrix. If the control law:

$$\mathbf{u}(n)=\mathbf{Fy}(n)+\mathbf{v}(n) \quad (2)$$

(where $\mathbf{F}$ is an $\nu_1 x \nu_2$ constant matrix and $\mathbf{v}(n)$ is the reference input vector) is applied to the system (1.1) and (1.2), the state equations of the system will be:

$$\mathbf{x}(n+\mathbf{1})=\left(\mathbf{A}+\mathbf{BF}(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{C}\right)\mathbf{x}(n)+ +\left(\left(\mathbf{BF}(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{D}+\mathbf{BF}\right)\right)\mathbf{v}(n) \quad (3.1)$$

$$\mathbf{y}(n)=(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{Cx}(n)+(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{Dv}(n) \quad (3.2)$$

It is known that the position and the structure of eigenvalues of (3.1) and (3.2) can be fully described by the invariant polynomials of the matrix $\left[\mathbf{I}z-\mathbf{A}-\mathbf{BFC}\right]$. This is called the eigenstructure assignment problem by constant output feedback.

The eigenstructure assignment problem by a linear state-variable feedback control law has been studied in [2], [3], [4], and [5].

In this paper, the eigenstructure assignment problem by constant output feedback for linear multivariable completely reachable and observable systems is studied.

2 Problem Statement

We consider the linear time-invariant discrete-time, completely reachable and observable system (1.1), (1.2) with $rank\left[\mathbf{C}*(\mathbf{I}z-\mathbf{A})^{-1}\mathbf{B}\right]=r\le min(m,p)$.

Let $\mathbf{c_i}(z)$, for i=1,2,...,q, $q\le r$ be arbitrary monic polynomials over $\Re[z]$ ($\Re[z]$ denotes the ring of polynomials with coefficients in *R*) that satisfy the foolowing conditions:

$$\mathbf{c}_{i+1}(z)/\mathbf{c}_i(z) \text{ for i=1,2,...,q-1} \quad (4)$$

$$\sum_{i=1}^{q} deg\,\mathbf{c}_i(z)=n \quad (5)$$

$$\sum_{i=1}^{t} deg\,\mathbf{c}_i(z)\ge\sum_{i=1}^{t}\mathbf{w}_i \text{ for t=1,...,q} \quad (6)$$

where $\mathbf{w}_i$ for i=1,2,…,m are the reachability indices of system (11), (1.2).

The eigenstructure assignment problem by constant output feedback can be stated as follows. Find the control law as in (2) such that the polynomial matrix $[\mathbf{I}z-\mathbf{A}-\mathbf{BFC}]$ has invariant polynomials $\mathbf{c}_i(z)$ for i=1,2,...,q completed by $\mathbf{c}_{q+1}(z)=...=\mathbf{c}_n(z)=1$

3 Basic Concepts

For the $p\times q$ polynomial matrix $\mathbf{A}(z)$, we denote as v_i the degree of *ith* row of $\mathbf{A}(z)$ and as w_i the degree of *ith* column of $\mathbf{A}(z)$. The matrix $\mathbf{A}(z)$ is said to be row reduced if its highest row degree coefficient matrix $\mathbf{A}_{hr}=\lim_{z\longrightarrow\infty} diag\left[z^{-v}1,z^{-v}2,\dots,z^{-v}p\right]$ has rank p. Similarly $\mathbf{A}(z)$ is column reduced if its highest column degree coefficient matrix $\mathbf{A}_{hc}=\lim_{z\longrightarrow\infty} diag\left[z^{-w}1,\dots,z^{-w}q\right]$ has rank q. If the rows of $\mathbf{A}(z)$ are arranged so that $v_i\ge v_j$, for i<j, then $\mathbf{A}(z)$ is *row degree ordered*, if $w_i\ge w_j$ for i<j is *column degree ordered*.

Two polynomial matrices $\mathbf{A}(z)$ and $\mathbf{B}(z)$, $p\times q$ and $p\times m$ respectively, are defined to be *relatively left prime* over $\Re[z]$ if there exist polynomial matrices $\mathbf{X}_1(z)$ and $\mathbf{Y}_1(z)$ such that $\mathbf{A}(z)\mathbf{X}_1(z)+\mathbf{B}(z)\mathbf{Y}_1(z)=\mathbf{I}$. Similarly two polynomial matrices $\mathbf{A}(z)$ and $\mathbf{B}(z)$, $q\times m$ and $p\times m$ respectively, are defined to be *relatively right prime* over $\Re[z]$ if there exist the polynomial matrices $\mathbf{X}_2(z)$ and $\mathbf{Y}_2(z)$ such that:

$$\mathbf{X}_2(z)\mathbf{A}(z)+\mathbf{Y}_2(z)\mathbf{B}(z)=\mathbf{I}$$

Definition 1. Polynomial matrices $\mathbf{D}_R(z)$ and $\mathbf{N}_R(z)$ such that:

(a). $\mathbf{C}(\mathbf{I}z-\mathbf{A})^{-1}\mathbf{B}=\mathbf{N}_R(z)\mathbf{D}_R^{-1}(z)$

(b). $\mathbf{D}_R(z)$ and $\mathbf{N}_R(z)$ are relatively right prime

(c). $\begin{bmatrix}\mathbf{D}_R(z)\\ \mathbf{N}_R(z)\end{bmatrix}$ is column reduced and column degree ordered

are said to form a *standard right matrix fraction description* of the system (1.1) and (1.2).

Definition 2. Polynomial matrices $\mathbf{D}_L(z)$ and $\mathbf{N}_L(z)$ such that:

(a). $\mathbf{C}(\mathbf{I}z-\mathbf{A})^{-1}\mathbf{B}=\mathbf{N}_L(z)\mathbf{D}_L^{-1}(z)$

(b). $\mathbf{D}_L(z)$ and $\mathbf{N}_L(z)$ are relatively left prime

(c). $[\mathbf{D}_L(z),\mathbf{N}_L(z)]$ is row reduced and row degree ordered

are said to form a *standard left matrix fraction description* of system (1.1) and (1.2).

The *reachability indices* of the system (1.1), (1.2) are defined to be the column degrees of any standard right matrix fraction description of the system (1.1), (1.2).

The observability indices of the system (1.1), (1.2) are defined to be the row degrees of any standard right matrix fraction description of the system (1.1), (1.2).

The polynomial matrices $\mathbf{N}_R(z)$ and $\mathbf{N}_L(z)$ are equivalent over $\Re[z]$ and the diagonal elements of its Smith form describes the position and the structure of the finite zeros of the system (1.1) and (1.2).

4 Problem Solution

We desire the invariants polynomials of $\mathbf{zI}-\mathbf{A}$ to be $P_i(z)$. To this end, the matrix $\mathbf{F}$ should be determinated in order to have:

$$\mathbf{zI}-\left(\mathbf{A}+\mathbf{BF}(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{C}\right)=\mathbf{U}(z)\mathbf{S}(z)\mathbf{W}(z) \qquad (7)$$

where $det\,\mathbf{U}(z)$ and $det\,\mathbf{W}(z)$ are constant or:

$$\frac{\partial}{\partial z}\left(det\,\mathbf{U}(z)\right)=0 \qquad (8.1)$$

$$\frac{\partial}{\partial z}\left(det\,\mathbf{W}(z)\right)=0 \qquad (8.2)$$

and:

$$\mathbf{S}(z)=\begin{bmatrix} P_1(z) & 0 & & 0 \\ 0 & P_2(z) & & 0 \\ & & \ddots & \\ 0 & 0 & & P_v(z) \end{bmatrix} \qquad (9)$$

or in a more compact notation:

$$\mathbf{S}(z)=diag\left\{P_1(z),P_2(z),\ldots,P_v(z)\right\} \qquad (10)$$

where $P_1(z)$, $P_2(z)$,..., $P_v(z)$ known polynomials for which $P_i(z)/\,P_{i+1}(z)$.

In general (7) can have or not a (theoritical) solution with respect to $\mathbf{F}$. Even if a solution $\mathbf{F}$ of (7) exists, several computational difficulties exist. To overcome these difficulties, the following minimization procedure is proposed:

$$min\frac{1}{2\pi j}\oint_{|z|=1}\left\|\mathbf{zI}-\left(\mathbf{A}+\mathbf{BF}(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{C}\right)-\right.$$
$$\left.-\mathbf{U}(z)\mathbf{S}(z)\mathbf{W}(z)\right\|_2^2 dz \qquad (11)$$

under the constraints:

$$\frac{\partial}{\partial z}\left(det\,\mathbf{U}(z)\right)=0 \qquad (11.1)$$

$$\frac{\partial}{\partial z}\left(det\,\mathbf{W}(z)\right)=0 \qquad (11.2)$$

Based on the well known residues calculus, our problem is reduced to the following minimization problem:

$$min\ \frac{1}{2\pi j}\sum_{k=0}^{v}\left\|\mathbf{zI}-\left(\mathbf{A}+\mathbf{BF}(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{C}\right)\right.$$
$$\left.-\mathbf{U}(z)\mathbf{S}(z)\mathbf{W}(z)\right\|_2^2 \qquad (12)$$

under the constraints:

$$\frac{\partial}{\partial z}\left(det\,\mathbf{U}(z)\right)=0 \qquad (13.1)$$

$$\frac{\partial}{\partial z}\left(det\,\mathbf{W}(z)\right)=0 \qquad (13.2)$$

Using the method of Lagrange multipliers we have:

$$min\left(\sum_{k=0}^{v}\left\|\mathbf{zI}-\left(\mathbf{A}+\mathbf{BF}(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{C}\right)\right.\right.$$
$$\left.-\mathbf{U}(z)\mathbf{S}(z)\mathbf{W}(z)\right\|_2^2+$$
$$\left.+\lambda_1\frac{\partial}{\partial z}\left(det\,\mathbf{U}(z)\right)+\lambda_1\frac{\partial}{\partial z}\left(det\,\mathbf{U}(z)\right)\right\} \qquad (14)$$

where the minimization takes place over

$\mathbf{F}_{ij}\ ,\mathbf{U}_{ij},\mathbf{U}_{ij}^{'},\mathbf{W}_{ij},\mathbf{W}_{ij}^{'},\lambda_1,\lambda_2$

In the sequel we can assume that:

$$\mathbf{U}(z)=\begin{bmatrix}\mathbf{U}_{\mathbf{11}}+\mathbf{U}_{11}^{'}z & \mathbf{U}_{\mathbf{1}n}+\mathbf{U}_{1n}^{'}z\\ \mathbf{U}_{n\mathbf{1}}+\mathbf{U}_{n1}^{'}z & \mathbf{U}_{nn}+\mathbf{U}_{n1}^{'}z\end{bmatrix} \qquad (15)$$

$$\mathbf{W}(z)=\begin{bmatrix}\mathbf{W}_{\mathbf{11}}+\mathbf{W}_{11}^{'}z & \mathbf{W}_{\mathbf{1}n}+\mathbf{W}_{1n}^{'}z\\ \mathbf{W}_{n\mathbf{1}}+\mathbf{W}_{n1}^{'}z & \mathbf{W}_{nn}+\mathbf{W}_{n1}^{'}z\end{bmatrix} \qquad (16)$$

Therefore, our problem is:

$$min\left(\sum_{k=0}\left\|\mathbf{zI}-\left(\mathbf{A}+\mathbf{BF}(\mathbf{I}-\mathbf{DF})^{-1}\mathbf{C}\right)\right.\right.$$
$$\left.-\mathbf{U}(z)\mathbf{S}(z)\mathbf{W}(z)\right\|_2^2+$$
$$\left.+\lambda_1\frac{\partial}{\partial z}\left(det\,\mathbf{U}(z)\right)+\lambda_1\frac{\partial}{\partial z}\left(det\,\mathbf{U}(z)\right)\right\}$$
$$\mathbf{F}_{ij},\mathbf{U}_{ij},\mathbf{U}_{ij}^{'},\mathbf{W}_{ij},\mathbf{W}_{ij}^{'},\lambda_1,\lambda_2 \qquad (17)$$

5 Conclusion

In this note, the problem of eigenstructure assignment by constant output feedback for linear time-invariant systems via appopriate minimization is examined and solved. The problem is reduced to the minimization of the distance of the desired characteristic matrix from the characteristic matrix of the under output feedback control law system at the unit circle. The method can also be extended for other classes of systems like multidimensional systems, multirate systems, hybrid systems, large-scale systems etc.

References:

[1] H.H.Rosenbrock, State-space and Multivariable Theory, Willey, New York, 1970.

[2] R. E. Kalman, “Kronecker invariants and feedback control”, In Ordinary Differencial Equations (L. Weiss Ed.)

[3] D.S.Flamm, “A new proof of Rosenbrock’s theorem on pole assignment” IEEE Tran. Automat. Contr., vol. AC-25

[4] B. W. Dickinson, “On the fundamental theorem of state variable feedback,” IEEE Trans. Automat. Contr., Vol. AC-19 pp.557-579, 1980

[5] V. Kucera, “Assigning the invariant factors by feedback”, Kybernetika, Vol. 17, pp.118-127, 1980

[6] N. Mastorakis(ed.), "Recent Advances in Circuits and Systems", World Scientific Pu. Co. 1998.

[7] K. H. Kiritsis, Linear Control by Constant Output Feedback, PhD Thesis,

National Technical University of Athens, Department of Electrical Engineering, Athens 1998.

[8] A.C. Pugh and P.A. Ratchliffe, “Infinite frequency intepretations of minimal bases”, Int.J.Contr., Vol.32, No.4, pp.551-558, 1980