Stability Margin of Two-Dimensional Continuous Systems[1]

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Abstract:- In this paper, the margin of stability of 2-D (Two-Dimensional) continuous systems is considered. In particular, the definition of the stability margins for a 2-D discrete system is extended to a 2-D continuous one. Two methods to compute the stability margin of 2-D continuous systems are presented. Illustrative examples are provided.

Keywords: Stability, Stability Margin, 2-D systems

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1 Introduction

Much interest has been shown during recent years in 2-D continuous systems ([1]÷[10], [17], [30], [41]÷[45]) for several reasons: In the design of 2-D and *m*-D (*m*>2) discrete filters, the corresponding analog filters play an important role. In particular, it is possible using appropriate transformations to obtain the desirable 2-D discrete filter from the corresponding analog (2-D) filter [2]÷[9], [41]. On the other hand, in the study of Distributed Parameter Systems (DPS) which are described by Partial Differential Equations (PDEs), each PDE actually corresponds to an *m*-D continuous system. So, for networks which include transmission lines as well as passive lumped elements, for networks containing semiconductor elements, for acoustic filters, the description with 2-dimensional continuous systems is necessary as one can see in [1], [4], [7], [8]. A third reason is the need of the introduction of the 2-D continuous systems theory in Control Systems whose coefficients are functions of parameters as well as in Systems whose inputs and outputs are functions of a time variable and a discrete spatial variable [8], [42]÷[44]. For these reasons, there exists an importance of the subject of the *m*-D continuous systems from a practical point of view ([1÷10], [17], [23], [30], [41]÷[45]).

Stability testing of the 2-D and *m*-D (*m*>2) continuous systems is of much importance [1÷10], [30]. Let a Linear Shift Invariant 2-D continuous system be described by the transfer function

$$G(s_1,s_2)=\frac{P(s_1,s_2)}{F(s_1,s_2)} \qquad (1)$$

where $P(s_1,s_2)$ and $F(s_1,s_2)$ are coprime polynomials in the independent complex variables s_1 and s_2, It is assumed that there are no nonessential singularities of the second kind on the double imaginary axis, i.e. there are no points s_1,s_2 with $s_1 = j\omega_1$ or ∞, $s_2 = j\omega_2$ or ∞ such that $P(s_1,s_2)=F(s_1,s_2)=0$.

The system (1) is (Hurwitz) stable if and only if

$$F(s_1,1)\neq 0, \qquad \text{for} \quad Re\{s_1\}\geq 0 \ \text{ or } \ s_1=\infty \qquad (2.1)$$

and

$$F(j\omega_1,s_2)\neq 0, \qquad \text{for } \left(Re\{s_2\}\geq 0 \text{ or } s_2=\infty\right) \text{ and } -\infty\leq\omega_1\leq\infty \qquad (2.2)$$

Additionally, the polynomial $F(s_1,s_2)$ is said to be *Hurwitz Polynomial* if and only if (2.1) and (2.2) are fulfilled. Condition (2.1) is relatively easy to check using any 1-D stability test. Checking condition (2.2) is a more difficult task.

There exist several algebraic methods for testing the stability of 2-D continuous systems or, equivalently, checking the Hurwitz character of 2-D polynomials [30], [28], [29]. Among them are the table form as advanced by Siljak [13], the determinant method (Anderson-Jury [14]), the inner method [16], [21] as advanced in [24], and the Zeheb-Walach method [12]. Another somewhat tedious approach is the method of the bilinear transformation. However, as pointed out by Goodman [20] and Jury and Bauer [22] the bilinear transformation can cause some difficulties in the stability tests because of the presence of nonessential singularities of the second kind and the value of the function at infinity [18], [19].

In the study of 2-D continuous systems, we are interested not only in whether the system is stable but also whether the system will remain stable in the presence of system parameter deviations.

For this reason and analogously to 2-D discrete systems [31], the following definition is introduced:

Definition 1: Given a 2-D continuous system described by the transfer function (1), we call stability margin σ_1 the greater non positive real number for which $F(s_1+\sigma_1,s_2)$ is a *Hurwitz Polynomial*.

Similarly the following two definitions are stated.

Definition 2: Given a 2-D continuous system described by the transfer function (1), we call stability margin σ_2 the greater non positive real number for which $F(s_1,s_2+\sigma_2)$ is a *Hurwitz Polynomial*.

Definition 3: Given a 2-D continuous system described by the transfer function (1), we call stability margin σ the greater non positive real number for which $F(s_1+\sigma, s_2+\sigma)$ is a *Hurwitz Polynomial*.

Note that the special case where the stable system has nonessential singularities of the second kind for some $s_1 = j\omega_1$ or ∞ and $s_2 = j\omega_2$ or ∞ is excluded, since all three stability margins will be zero.

In 2-D discrete systems these definitions were given in [31], while several methods for the evaluation of stability margins already exist [31]÷[38]. In this paper, the analogous methods in 2-D continuous case are derived the applicability and effectiveness of which are illustrated by some examples.

2 Computation of the stability margins for 2-D continuous systems

A. Hermite Matrix Method

In this paragraph, a method of computing the stability margin of 2-D continuous systems is presented. The method is based on checking the positive definiteness of the Hermite matrix of the characteristic polynomial $F(s_1,s_2)$ of a stable system described by (1). For a 2-D continuous system we recall that the Hermite matrix $H_1(\omega_1)$ associated with $F(j\omega_1, s_2)$ is positive definite where $-\infty \le \omega_1 \le \infty$ and $Re\{s_2\} \ge 0$ or $s_2 = \infty$ [30].

Considering the Hermite matrix $H_1(\omega_1,\sigma_1)$ associated with $F(j\omega_1+\sigma_1, s_2)$, we obtain that in the "limit point" of the maximum value of σ_1 the Hermite matrix $H_1(\omega_1,\sigma_1)$ will be positive semidefinite ($-\infty \le \omega_1 \le \infty$, $Re\{s_2\} \ge 0$ or $s_2 = \infty$). This implies that for this point the matrix $H_1(\omega_1,\sigma_1)$ is singular. Thus the computation of σ_1 can be achieved by solving the following optimization problem

$$\max_{\sigma_1 \le 0} \sigma_1 \tag{3}$$

under the constraint

$$\det H_1(\omega_1,\sigma_1)=0 \tag{4}$$

where $H_1(\omega_1,\sigma_1)$ is the Hermitian matrix associated with $F(j\omega_1+\sigma_1, s_2)$. By interchanging the roles of the variables s_1 and s_2, a completely analogous method for the computation of σ_2 is obtained. Moreover, for the computation of σ we demand

$$\max_{\sigma \le 0} \sigma \tag{5}$$

under the constraint

$$\det H(\omega_1,\sigma)=0 \tag{6}$$

where $H(\omega_1,\sigma)$ is the Hermitian matrix associated with $F(j\omega_1+\sigma, s_2+\sigma)$. The following example illustrates the implementation of this method.

Example 1: Consider the characteristic polynomial of a 2-D (continuous) system

$$F(s_1,s_2) = 7 + s_1 + 2s_2 + 2s_1s_2 \tag{7}$$

It is always assumed that the corresponding 2-D system has no nonessential singularities of the second kind. Obviously, condition (2.1) holds while condition (2.2) can be easily checked via the positive definiteness of the Hermitian matrix which is $H_1(\omega_1)=14+2\omega_1^2$. Therefore $F(s_1,s_2)$ is a *Hurwitz Polynomial*. For the computation of the stability margin σ_1, one forms the Hermitian determinant (i.e. the determinant of the Hermitian matrix) of $F(j\omega_1+\sigma_1, s_2)$. This is

$$H_1(\omega_1,\sigma_1) = (2+2\sigma_1)(7+\sigma_1) + 2\omega_1^2 \tag{8}$$

The *maximum* value of σ_1 ($\sigma_1 \le 0$) for which $H_1(\omega_1,\sigma_1)$ is nullified for some ω_1 is obviously -1. Therefore the stability margin σ_1 is -1. Quite analogously one finds that $\sigma_2=-\frac{1}{2}$. In order to evaluate the stability margin σ, we form the Hermitian determinant associated with $F(j\omega_1+\sigma, s_2+\sigma)$. This can be found as

$$H(\omega_1,\sigma)=(2+2\sigma)(7+3\sigma+2\sigma^2)+2\omega_1^2(1+2\sigma) \tag{9}$$

We obtain that for $\sigma \to -\frac{1}{2}^{+}$, $H(\omega_1,\sigma)$ is positive, but for $\sigma \to -\frac{1}{2}^{-}$ and $\omega_1 \to \infty$, $H(\omega_1,\sigma)$ is negative. So, if $\sigma \to -\frac{1}{2}^{-}$ and one allows $\omega_1=\sqrt{-\frac{(2+2\sigma)(7+3\sigma+2\sigma^2)}{2(1+2\sigma)}}$ $(\longrightarrow \infty)$, then $H(\omega_1,\sigma)$=0. Therefore $\sigma=-\frac{1}{2}$.

Example 2: Consider the general first order characteristic polynomial of a 2-D (continuous) system

$$F(s_1,s_2)=1+as_1+bs_2+cs_1s_2 \tag{10}$$

where $a,b,c>0$. By verifying the same conditions as in Example 1, one can easily see that this a *Hurwitz Polynomial*. Similarly one finds

$$H_1(\omega_1,\sigma_1)=(b+c\cdot\sigma_1)(1+a\cdot\sigma_1)+a\cdot c\cdot\omega_1^2 \tag{11}$$

So, the *maximum* non-positive value of σ_1 for which $H_1(\omega_1,\sigma_1)$ is nullified for some ω_1 is:

$$\sigma_1=max\left\{-\frac{b}{c},-\frac{1}{a}\right\} \tag{12}$$

Consequently, interchanging the variables s_1 and s_2 one evaluates

$$\sigma_2=max\left\{-\frac{a}{c},-\frac{1}{b}\right\} \tag{13}$$

To obtain σ, we form the Hermitian determinant associated with $F(j\omega_1+\sigma, s_2+\sigma)$.

$$H(\omega_1,\sigma)=(b+c\cdot\sigma)\left(1+(a+b)\cdot\sigma+c\cdot\sigma^2\right)+\omega_1^2\cdot c\cdot(a+c\cdot\sigma) \tag{14}$$

We assert that σ is the maximum of the real roots of the polynomials (in σ) $(b+c\cdot\sigma)$, $\left(1+(a+b)\cdot\sigma+c\cdot\sigma^2\right)$, $(a+c\cdot\sigma)$. To prove this, we suppose that $-\frac{a}{c}$ is greater than the (real) roots of the polynomials $(b+c\cdot\sigma)$, $\left(1+(a+b)\cdot\sigma+c\cdot\sigma^2\right)$. Then if $\sigma \to -\frac{a}{c}^{-}$ and $\omega_1=\sqrt{-\frac{(b+c\sigma)\left(1+(a+b)\sigma+c\sigma^2\right)}{c(a+c\sigma)}}$,$(\to \infty$ since $(b+c\sigma)\left(1+(a+b)\sigma+c\sigma^2\right)$>0 for $\sigma \to -\frac{a}{c}^{-}$ $(a,b,c>0)$) we obtain that $H(\omega_1,\sigma)$=0. On the other hand, if one root σ_r of the polynomials $(b+c\cdot\sigma)$, $\left(1+(a+b)\cdot\sigma+c\cdot\sigma^2\right)$ is greater than $-\frac{a}{c}$ then for $\sigma=\sigma_r$ and $\omega_1=0$ we find $H(\omega_1,\sigma)$=0. Hence,

$$\sigma = \begin{cases} max\left(\frac{-(a+b)+\sqrt{(a+b)^2-4c}}{2}, -\frac{b}{c}, -\frac{a}{c} \right) & \text{if } \ (a - \\ max\left(-\frac{b}{c}, -\frac{a}{c} \right) & \end{cases}$$

It is worth noticing the symmetry between *a* and *b*.

B. <u>Geometrical Method</u>

Let $F(s_1, s_2)$ be the characteristic polynomial of a 2-D stable system.

$$F(s_1, s_2) = \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} b(i_1, i_2) s_1^{i_1} s_2^{i_2} \qquad (16)$$

For the computation of the stability margin σ_1, symmetrically to condition (2.2), we consider the relation

$$F(s_1, j\omega_2) = 0 \qquad (17)$$

If ω_2 vary along the typical Nyquist curve (Fig.1), then the solution of the (1-dimensional with respect to s_1) polynomial equation (17) provides N_1 branches for the complex variable s_1. For the sake of simplicity, in Fig.2, we sketched only one branch. We also note that the function $F(s_1, j\omega_2)$ is differentiable respect to s_1 and ω_2. This implies that all the derivatives in the following relations will be meaningful. Because of the stability of the considered system, all the branches will lie in the complex open left semiplane.

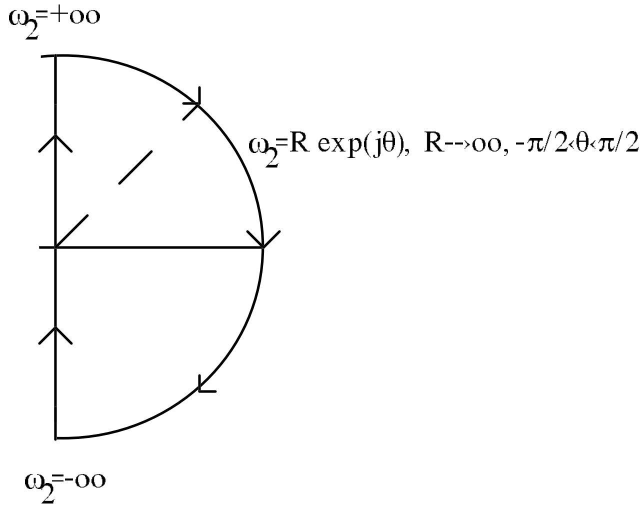

Fig.1: Nyquist curve of the complex variable $s_2 = j\omega_2$

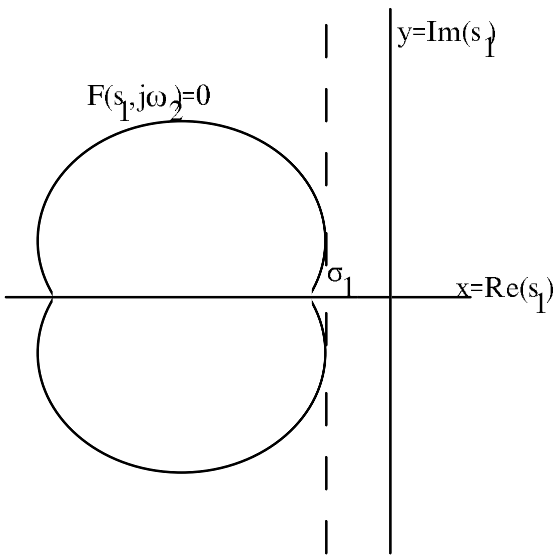

Fig.2: A geometrical interpretation of Equation (17)

Differentiating (17) along the considered branch of s_1, one obtains

$$\frac{\partial F(s_1, j\omega_2)}{\partial s_1} ds_1 + \frac{\partial F(s_1, j\omega_2)}{\partial \omega_2} d\omega_2 = 0 \qquad (18)$$

Since the solution of the (1-Dimensional) Eq. (17) with respect to s_1 is always possible, we have that

$\frac{\partial F(s_1, j\omega_2)}{\partial s_1} \neq 0$ [26], [27]. Therefore, (18) renders,

$$\frac{\partial s_1}{\partial \omega_2} = -\frac{\frac{\partial F(s_1, j\omega_2)}{\partial \omega_2}}{\frac{\partial F(s_1, j\omega_2)}{\partial s_1}} \qquad (19)$$

Equation (19) guarantees the existence of the (geometrical) tangent of all the branches of (17) at any point. Moreover, at the closest point of all these branches to the imaginary axis, one observes that the tangent will be parallel to the imaginary axis (Fig.2). Therefore a simple necessary condition for the stability margin σ_1 will be

$$Arg\left[\frac{\partial s_1}{\partial \omega_2}\right] = \pm\frac{\pi}{2} \qquad (20)$$

Taking into account that $s_2 = j\omega_2$, one rewrites (19) as

$$\frac{\partial s_1}{\partial \omega_2} = -j\left.\frac{\frac{\partial F(s_1, s_2)}{\partial s_2}}{\frac{\partial F(s_1, s_2)}{\partial s_1}}\right|_{s_2 = j\omega_2} \qquad (21)$$

Therefore

$$Arg\left[\left.\frac{\frac{\partial F(s_1, s_2)}{\partial s_2}}{\frac{\partial F(s_1, s_2)}{\partial s_1}}\right|_{s_2 = j\omega_2}\right] = 0 \text{ or } \pi \qquad (22)$$

It is more convenient to write Equation (22) in the form

$$\left.\left(\frac{\partial F(s_1, s_2)}{\partial s_1} \times \frac{\partial F(s_1, s_2)}{\partial s_2}\right)\right|_{s_2 = j\omega_2} = 0 \qquad (23)$$

where the outer product $(\times)$ of the complex numbers $x_1 + jy_1$ and $x_2 + jy_2$ is defined as the algebraic value of the outer product of the corresponding vectors (x_1, y_1) and (x_2, y_2) i.e. $x_1y_2 - x_2y_1$. We could also have written Equation (23) in a more general form

$$\left.\left(\frac{\partial F(s_1, s_2)}{\partial s_1} // \frac{\partial F(s_1, s_2)}{\partial s_2}\right)\right|_{s_2 = j\omega_2} \qquad (24)$$

where // denotes the parallelness of the corresponding vectors.

Thus, the stability margin σ_1 is evaluated by the solution of (17) and (23) (or (24)). These two equations can readily be put in the form of three polynomial equations in the three independent variables x, y, ω_2, where $x = Re\{s_1\}$ and $y = Im\{s_1\}$. Their common solution(s) can be found, if desired, using the resultant method of these polynomials [16], [39]. In the sequel, one must select as σ_1 the maximum x of all the solutions (x, y, ω_2).

A similar method can also be formulated to derive the stability margin σ_2. For the stability margin σ, instead of (17) and (23) we get the equations

$$F(s_1,s_2)\Big|_{\substack{s_1=x+jy\\ s_2=x+j\omega_2}}=0 \qquad (25)$$

and

$$\left(\frac{\partial F(s_1,s_2)}{\partial s_1}\times\frac{\partial F(s_1,s_2)}{\partial s_2}\right)\Bigg|_{\substack{s_1=x+jy\\ s_2=x+j\omega_2}}=0 \qquad (26)$$

These equations can also be put in the form of three polynomial equations in x,y,ω_2. Their common solution(s) can be derived if desired, from the resultant method. Furthermore, the stability margin σ is evaluated as the maximum x of all the solutions (x,y,ω_2).

Example 3: To demonstrate the method, we consider the same characteristic polynomial as in Example 2.

$$F(s_1,s_2)=1+as_1+bs_2+cs_1s_2$$

where $a,b,c>0$. Eq.(17) and (23) yield

$$1+ax-cy\omega_2=0 \qquad (27)$$

$$ay+(b+cx)\omega_2=0 \qquad (28)$$

$$acy-c\omega_2(b+cx)=0 \qquad (29)$$

where $x=Re\{s_1\}$ and $y=Im\{s_1\}$.

If $\omega_2\neq\infty$, then solving (28) with respect to $(b+cx)\omega_2$ and substituting into (29), one finds $y=0$. Then, from (27), $x=-\frac{1}{a}$. If $\omega_2\to\infty$, (28) implies that $y\to\infty$ or $b+cx=0$. However, if $y\to\infty$, from (27) we obtain $x\to\infty$ which is not accepted. Therefore $b+cx=0$ or $x=-\frac{b}{c}$ and $y=\frac{c-ab}{c^2\omega_2}\xrightarrow{\omega_2\to\infty}0$. In this case, (28) and (29) hold considering limits in their left hand side. Therefore

$$\sigma_1=max\left\{-\frac{b}{c},-\frac{1}{a}\right\} \qquad (30)$$

Similarly, interchanging the variables s_1 and s_2

$$\sigma_2=max\left\{-\frac{a}{c},-\frac{1}{b}\right\} \qquad (31)$$

For the stability margin σ, Eq.(25) and (26) render

$$1+(a+b)x+cx^2-c\omega_2y=0 \qquad (32)$$

$$(a+cx)y+(b+cx)\omega_2=0 \qquad (33)$$

$$(a+cx)cy-c\omega_2(b+cx)=0 \qquad (34)$$

From (32), $y=\frac{1+(a+b)x+cx^2}{c\omega_2}$. Multiplying (33) by $c\omega_2$ and substituting the term $c\omega_2y$ from (32), we get

$$(a+cx)\left(1+(a+b)x+cx^2\right)+(b+cx)c\omega_2^2=0 \quad (35)$$

Similarly, (34) yields

$$(a+cx)\left(1+(a+b)x+cx^2\right)-(b+cx)c\omega_2^2=0 \quad (36)$$

Obviously, from (35) and (36) if $\omega_2\neq\infty$, one obtains $(a+cx)\left(1+(a+b)x+cx^2\right)=0$ as well as $\omega_2=0$. On the other side, if $-\frac{b}{c}$ is greater than any real root of $(a+cx)\left(1+(a+b)x+cx^2\right)$ and if $\omega_2\to\infty$, from (35), one finds $x\to-\frac{b^-}{c}$. So, if we let $x\to-\frac{b^-}{c}$, then we obtain

$\omega_2 = \sqrt{-\frac{(a+cx)\left(1+(a+b)x+cx^2\right)}{b+cx}} \to \infty$. In this case, (33) and (34) hold considering limits in the left hand side. Evidently, σ is the maximum of all *x*, i.e. the maximum of $-\frac{b}{c}$ and the real roots of $(a+cx)\left(1+(a+b)x+cx^2\right)$. Thus, the stability margin σ is given by Eq.(15).

Remark 1: All the above results are extended to the *m*-D (*m*>2) continuous case after simple modifications.

Remark 2: An interesting generalization of the Definitions of $\sigma_1, \sigma_2, \sigma$ could be the following:

Definition of the stability margin σ *with weights* k, λ $(k+\lambda=1,\ k \geq 0,\ \lambda \geq 0)$: Given a 2-D continuous system described by the transfer function (1), we call stability margin σ the greater non positive real number for which $F(s_1+k\sigma, s_2+\lambda\sigma)$ is a *Hurwitz Polynomial* Modifying anyone of the above methods, one can easily derive two appropriate algorithms for evaluating the stability margin σ with weights k, λ.

3 Conclusions

In many engineering applications (study of distributed parameter systems, design of 2-D discrete filters via appropriate transformations of 2-D continuous ones, study of automatic control systems whose coefficients are functions of parameters and study of systems whose inputs and outputs are functions of a time variable and a discrete spatial variable) the class of 2-D continuous systems has been proved as a very useful class. In the study of 2-D continuous systems, we are interested not only in whether the system is stable but also how far is the system from the region of instability. A measure of this is the so called stability margin.

Two methods for computing the stability margin of 2-D continuous systems have been proposed in this paper. The first method is based on a constrained optimization of a real non-positive parameter. The second method is based on a geometrical consideration of the whole problem. Some examples are given which also show the equivalence of the methods.

One could obtain that the first method is simpler and easy to understand but an optimization problem is needed to be solved. Because of the use of optimization, this method does not always yield an exact value. From this point of view, the geometrical method seems to be better, even if we have to select the solution with the *max*(*x*).

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