Polynomial System Triangularization with the Dixon Resultant

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Abstract: - Solving polynomial systems has been of great interest to both applied and theoretical scientists for quite some time. Recently this interest has been reinforced by numerous demands from diverse areas such as robotics, threat analysis, computer graphics, automatic geometric theorem proving, and invariant theory. New ideas have emerged from computer algebra, most notably methods based on Groebner bases and on resultants. The former yield very satisfactory exact answers, but the computations are resource demanding. Various resultant methods have been the focus of recent attention, because they produce good results faster and more efficiently.

In 1908 Dixon introduced a new resultant that generalized Bezout's resultant. Dixon's resultant eliminated two variables out of a system of three polynomial equations, using basic properties of determinants. Dixon proved that if the system has a common solution, then the resultant vanishes. This is true provided that the polynomials have nonzero coefficients that are independent parameters. These are very strong restrictions that make the method impractical for solving most polynomial systems.

Recently, the Dixon resultant has been revisited and generalized by Kapur, Saxena, and Yang. The restrictions on the coefficients of the polynomial system in the generalized form are very mild and the method can be applied to almost any system of $n+1$ equations and n unknowns (parametric coefficients allowed). Subsequent work by Kapur and Saxena strongly suggests that this is the most efficient elimination method. Their comparisons include the Macaulay and the more recent Sparse resultant.

In this paper we address the question of how to actually solve a polynomial system by using the Dixon resultant. We describe a certain triangularization of the polynomial system based on the computation of several Dixon resultants. This method exploits the special structure of the matrix whose determinant is the Dixon polynomial. The new idea here is the following: instead of computing the determinant of this matrix directly, a fraction-free Gauss elimination is performed based on the well-known Gauss-Bareiss algorithm. When done properly, this elimination produces the principal minors of the original matrix on the main diagonal. Each of these minors yields a corresponding Dixon resultant with an increasing number of variables. Eventually, all these resultants form a triangular system whose solutions include – in general – the solutions of the original one.

Key-Words: - Polynomial Systems, Resultants, Dixon Resultant, Fraction-free Gauss Elimination

1 Introduction

In this paper we discuss how to transform a system of polynomial equations into one in triangular form by using the Dixon resultant [5]. More precisely, we use the recent generalization of this resultant by Kapur, Saxena, and Yang [7]. According to Kapur and Saxena [8], this generalized Dixon resultant is perhaps the most efficient way of eliminating a block of variables from a polynomial system. The comparison favors the Dixon resultant even over the popular Macaulay and Sparse resultants. We have partially verified these claims using our own implementations of the Dixon resultant in Maple [11] and Mathematica [12]. Our concern here is how to use this efficient elimination method to solve polynomial systems fast and in an automated way. Our new idea is to compute several Dixon resultants simultaneously by exploiting the special structure of the cancellation matrix that yields the Dixon polynomial. This is done by using a fraction-free Gauss elimination based on the Gauss-Bareiss algorithm [1]. A correct application of this algorithm yields an echelon form of the cancellation matrix with diagonal entries its principal minors. Each one of these minors defines in turn a Dixon resultant with one more variable than the preceding one. This results into a triangular system whose solutions, in general, include the solutions of the original system. Our first experiments with this new approach were very encouraging. The method seems to be fast and efficient. However, this is only a preliminary step to a pending more thorough investigation. Questions that need to answered are: What is the exact relation between the solution spaces of the original and the reduced system? How do the Dixon extraneous factors effect the original solution? How can one tell whether the original system has finitely many solutions, by solving the triangularized system?

The remaining of this paper is organized as follows. In Section 2 we introduce the classical Dixon resultant. Then in Section 3 we discuss the significant improvements made by Kapur, Saxena, and Yang. Section 4 is devoted to the process of triangularization and two examples. For simplicity, here we confine our discussion and examples mostly to three polynomial equations in two unknowns, although the method is applicable to n+1 equations in n unknowns, with possibly parametric coefficients. Finally, in Section 5 we discuss our conclusions so far and a direction for future work.

2 The Classical Dixon Resultant

In this section we introduce the Dixon resultant [1] which is a generalization to three equations of Bezout's elimination for two polynomial equations. We start with Cayley's formulation [2] of Bezout's method for solving a system of two polynomial equations. According to Kapur this approach goes back to Euler.

Let $f(x)$ and $g(x)$ be polynomials in x, let d be the maximum of the degrees of f and g, and let s be an auxiliary variable. The quantity

$$d(x,y) = \frac{1}{x-s}\begin{vmatrix} f(x) & g(x) \\ f(s) & g(s) \end{vmatrix}$$

is a symmetric polynomial in x and s of degree d–1 which we call the **Dixon polynomial** of f and g. Cayley (and Bezout in different notation) observed that: every common zero of f and g is also a zero of $d(x,y)$ for all values of s. Hence, at a common zero, each coefficient of s^i in $d(x,y)$ is identically zero. In matrix notation we have

$$M\begin{bmatrix} x^0 \\ x^1 \\ \vdots \\ x^{d-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where, the rows of the $d \times d$ matrix M consist of the coefficients of the s^i s. This yields a homogeneous system in new variables $v_1, \ldots, v_d$ corresponding to x^0, ..., x^{d-1} and equations corresponding to the coefficients of s^i.

$$M\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This system has nontrivial solutions if and only if its determinant D is zero. This determinant D is called the *Dixon resultant* of f and g. The matrix M is the *Dixon matrix*. The matrix whose determinant was used to compute the Dixon polynomial is called the *Cancellation matrix* of f and g.

So far we have seen that the vanishing of D is a necessary condition for the existence of a common zero of f and g.

Dixon generalized Cayley's approach to Bezout's method to systems of three polynomial equations in two unknowns. Starting with the system

$$f(x,y) = 0, \quad g(x,y) = 0, \quad h(x,y) = 0 \qquad (1)$$

we define d as follows

$$d(x,y,s,t) = \frac{1}{(x-s)(y-t)}\begin{vmatrix} f(x,y) & g(x,y) & h(x,y) \\ f(s,y) & g(s,y) & h(s,y) \\ f(s,t) & g(s,t) & h(s,t) \end{vmatrix} \qquad (2)$$

for auxiliary variables s and t. One again gets a homogeneous linear system just as before, by setting the coefficients of the power products $s^i t^j$ equal to zero. The corresponding determinant of the coefficient matrix is the Dixon resultant D and is free of the variables x and y.

To illustrate, let us consider the following example discussed in [6] and solved there by a combination of Sylvester resultants.

Example 1 Eliminate y and z from the following system that represents the intersection of two planes and a sphere.

$$x - az + b = 0$$
$$y - cz + d = 0$$
$$x^2 + y^2 + z^2 - R^2 = 0$$

Solution: The Dixon polynomial is given by

$$d(y,z,s,t) = \frac{1}{(y-s)(z-t)} \cdot$$

$$\begin{vmatrix} x-az+b & y-cz+d & x^2+y^2+z^2-R^2 \\ x-az+b & s-cz+d & x^2+s^2+z^2-R^2 \\ x-at+b & s-ct+d & x^2+s^2+t^2-R^2 \end{vmatrix}$$

$$= xt - azt + bt - ays - ady + cxy - aR^2$$
$$+ ax^2 - ads + bcy + cxs + bcs + xz + bz$$

Therefore, the Dixon matrix is

$$M = \begin{bmatrix} -aR^2 + ax^2 & b+x & cx+bc-ad \\ b+x & -a & 0 \\ cx+bc-ad & 0 & -a \end{bmatrix}$$

Its determinant is the Dixon resultant

$$D = a(-a^2R^2 + a^2x^2 + b^2 + 2bx + x^2 + c^2x^2$$
$$+ 2bc^2x - 2acdx + b^2c^2 - 2abcd + a^2d^2)$$

which is free of y and z.

Note In practice we compute the Dixon polynomial as the following determinant

$$d(x,y,s,t) = \begin{vmatrix} f(x,y) & g(x,y) & h(x,y) \\ f_1(x,y,s) & g_1(x,y,s) & h_1(x,y,s) \\ f_2(y,s,t) & g_2(y,s,t) & h_2(y,s,t) \end{vmatrix} \quad (3)$$

where

$$f_1(x,y,s) = \frac{f(s,y) - f(x,y)}{x-s}$$

$$f_2(y,s,t) = \frac{f(s,t) - f(s,y)}{y-t}$$

and g_1, g_2, h_1, h_2 are defined similarly. This works well when the polynomials are not very sparse and of high degree at the same time.

To understand what Dixon proved we need the definition of generic *n*degree polynomials.

Definition The polynomials $p_1, p_2, \ldots, p_{n+1}$ in variables $x_1, x_2, \ldots, x_n$ are *generic ndegree* if there exist nonnegative integers $k_1, k_2, \ldots, k_n$ such that for $1 \le j \le n+1$ we have

$$p_j = \sum_{i_1=0}^{k_1} \cdots \sum_{i_n=0}^{k_n} a_{j,i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$$

In other words, generic *n*degree polynomials have nonzero coefficients that are independent parameters. Dixon proved the following theorem.

Dixon's Theorem For a system with three generic *2*degree polynomials, the vanishing of the Dixon resultant D is a necessary condition for the existence of a common zero. Furthermore, D is not identically zero.

As noted in [9] Dixon's method and proofs easily generalize to a system of $n+1$ generic *n*degree polynomials in n unknowns.

This method can be applied to polynomials with symbolic coefficients and be used for a simultaneous elimination of a block of unknowns by using only one calculation. Often, even if the polynomials are not generic *n*degree the method may still yield a necessary condition for the existence of a common zero. These features, along with the relatively small size of the resulting determinants (compared with other resultant methods) makes the method *very* attractive.

However, if the polynomials are not generic and *n*degree, (which is typically the case in practice), then one may be faced with some serious problems. For general polynomial systems the vanishing of the Dixon resultant may *not* yield a necessary condition for the existence of a common zero. Also the Dixon matrix may be *singular*, which will make the Dixon resultant *identically zero*. This is exactly the case for the polynomial set

$$\begin{aligned} f &= xy + xz + x - z^2 - z + y^2 + y \\ g &= x^2 + xz - x + xy + yz - y \\ h &= x^2 + xy + 2x - xz - yz - 2z \end{aligned} \quad (4)$$

When we try to eliminate x and y from the Dixon polynomial

$$2xyz + 4tzy + 3zxs + 5zys - 3xys - 2yt - 2z$$
$$+ zy + zx + 2z^2 - 2zt + 2xyzs + 2txyz + 2tzxs$$
$$+ 2tzys + z^2x + z^2y + 2zy^2 + 2z^3x + 2z^3y$$
$$+ 2y^2zs - 2z^2xs + 2zxs^2 + 2zys^2 - txy + 2tzy^2$$
$$- tsy - 3txs - 4sy - 2xs + 2z^2s - 3sy^2 - s^2y$$
$$- 3xs^2 + 2tz^2 - ty^2 - 2z^2ys + 2tzx + 2tzs + 2zs^2$$

we get the following Dixon matrix

$$\begin{bmatrix} 2z^2-2 & 2z^3+z^2+z & 2z & 2z^3+z^2+z & 2z & 0 \\ 2z^2-2 & 4z-2 & 2z-1 & 2z & 2z-1 & 0 \\ 2z^2 & -2z^2+5z-4 & 2z-3 & -2z^2+3z-2 & 2z-3 & 0 \\ 2z & 2z-1 & 0 & 2z-3 & 0 & 0 \\ 2z & 2z-1 & 0 & 2z-3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

whose determinant is clearly zero.

Note that we could remove the zero rows and columns from a Dixon matrix but then the matrix may not even be square, so computing its determinant would be meaningless. In the next section we see how these important issues were resolved in [9].

3 The Kapur-Saxena-Young Approach

Kapur, Saxena and Yang resolved all the problems of the classical Dixon resultant, provided a certain precondition holds. Their work makes this method very practical and perhaps the elimination method of choice. Let us describe their main theorem and algorithm.

Suppose we have a system of n+1 polynomial equations in n variables such that the coefficients of the polynomials are themselves polynomials in a finite set of parameters. Let M be the Dixon matrix obtained as before, except that this time we also remove all zero columns and all zero rows from it. (So M can be rectangular.) Let M' be an echelon form matrix obtained from M by using elementary row operations *except* for scaling of rows. (Such a reduction is always possible.) Let D be the product of all pivots of M'. Under this notation we have the following theorem from [9].

Theorem 1 (Kapur-Saxena-Yang) If at least one of the columns of the Dixon matrix is not a linear combination of the remaining ones, then D=0 is a necessary condition for the existence of common zeros of the polynomial system.

This theorem yields a *simple* algorithm for obtaining the necessary condition D=0. We call D in the theorem as the Kapur-Saxena-Yang (KSY) Dixon resultant.

Let us refer to the assumption of the theorem about a column being a linear combination in the remaining ones as the *precondition*. We have the following algorithm.

Algorithm

Input: A set of polynomials, with numeric or parametric coefficients.

- Compute the Dixon matrix M. If the precondition holds continue.
- Row reduce M without scaling to row echelon form M'.
- Compute the product D of the pivots of M'.

Output: D, the Kapur-Saxena-Yang Dixon resultant. Its vanishing is a necessary condition for a solution of the given system.

The following theorem from [9] offers a rephrasing of the precondition of theorem 1.

Theorem 2

Let M be the Dixon matrix of a polynomial system and let $\mathbf{m}_1$, $\mathbf{m}_2$, ..., $\mathbf{m}_s$ be the columns of M. Let $\mathbf{w}=(w_1,w_2,..., w_s)^T$ be a solution of the homogeneous system

$$M\mathbf{w} = \mathbf{0} \Leftrightarrow w_1\mathbf{m}_1 + \cdots w_s\mathbf{m}_s = \mathbf{0}$$

Then precondition of theorem 1 is true for the ith column if and only if $w_i = 0$.

Testing for the validity of the precondition is usually, a simple test in practice. We personally use a probabilistic test *before* he row reduce the entire Dixon matrix M.

It is worth noting that the precondition is a rather mild assumption on the polynomial set. Most of the interesting examples seem to satisfy it, thus the KSY method applies.

To illustrate the power of the KSY method, let us now return to system (4) whose Dixon resultant turnes out to be identically zero. The KSY Dixon resultant is obtained by removing the zero rows and columns of Dixon matrix shown. Row reduction of the this matrix without scaling yields an echelon form matrix whose product of the leading entries is

$$D = 8z(z-1)(2z-1)(2z^2+3z-2)$$

This is not identically zero. In fact, solving D=0 for z yields all the values of z that lift to the exact rational solutions of the original system.

Many more examples as well as Maple and Mathematica programs of the Kapur, Saxena, and Yang approach can be found in [11] and [12].

4 A Triangularization Method

In this section we show how to combine the fraction-free algorithms with the Dixon resultant elimination to triangularize a system of polynomial equations. The solution set of the resulting system contains all the solutions of the original system, provided that the precondition of theorem 1 holds for each Dixon matrix in the process.

First, let us briefly review the Gauss-Bareiss algorithm. More details can be found in [1], [4], and [14]. This algorithm, due to Bareiss, is based on Sylvester's identity. The essence of the algorithm can be seen by considering a size three square matrix. The result of a divisionless Gauss elimination on matrix A

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is an upper triangular matrix

$$E(A) = \begin{bmatrix} a & b & c \\ 0 & ae - bd & af - cd \\ 0 & 0 & a\det(A) \end{bmatrix}$$

The final element has the factor a and that when removed yields det(A). In a case of a larger matrix this removal would yield the 3×3 principal minor. Furthermore with a larger matrix, *every* element below or to the right of the (3,3) entry would have this factor a at this stage. This factor is therefore removable from the entire active submatrix. The same comment applies for subsequent factors whose removal will leave all the principal minors on the diagonal of the original matrix.

Let us now see how to combine the fraction-free algorithm with the Dixon resultant elimination to triangularize a system of n polynomial equations in m variables, with m ³ n. Such a system we view as one in n-1 variables with parametric coefficients. For the simplicity of the illustration of the method, let us consider system (1) of Section 2. Instead of computing the Dixon polynomial d in one step, we apply a fraction-free elimination to the matrix A with determinant d from equation (3), Section 2.

$$A = \begin{vmatrix} f(x,y) & g(x,y) & h(x,y) \\ f_1(x,y,s) & g_1(x,y,s) & h_1(x,y,s) \\ f_2(y,s,t) & g_2(y,s,t) & h_2(y,s,t) \end{vmatrix}$$

Fraction-free elimination of A yields a matrix of the form

$$E(A) = \begin{bmatrix} f(x,y) & * & * \\ 0 & \det(A_1) & * \\ 0 & 0 & \det(A) \end{bmatrix}$$

where, $\det(A_1)$ is the 2×2 principal minor of A. Explicitly,

$$\det(A_1) = \frac{f(x,y)g(s,y) - g(x,y)f(s,y)}{x-s}$$

Note that if $\det(A_1)$ is considered as a polynomial in its own right its Dixon resultant will be free of x. Also det(A) has Dixon resultant free of x and y. So, we see that the diagonal elements of *E(A)* yield Dixon resultants that produce a system of the form f, g^x, h^{xy}, where g^x is free of x and h^{xy} is free of x and y. If the assumption of the theorem holds for each Dixon matrix then any solution of the original system is also a solution of f, g^x, h^{xy}. We call this method a *Dixon triangularization* of a polynomial system.

Example 2 Use Dixon triangularization to system (4) of Section 2.

Solution: The Dixon polynomials after the fraction-free Gauss elimination are

$$f = -y - z - 1$$

$$g = -ys + s + z^2 - y - st - s^2 + t - ty - y^2 - tz - z - xt - xs - x - xy$$

and

$$\begin{aligned} h = {} & -2z + 2z^2 s - ys^2 + 2s^2 z - 3xs^2 + 2ysz^2 \\ & + 2yzxs + 2zxs^2 - 2ty - ty^2 + 2tz^2 + 2y^2 z \\ & + 2txyz + 2tzxs + 2tyzs + 2y^2 zs - 2yz^2 s \\ & + 2z^3 y + 2z^3 x + 2z^2 + 2ty^2 z + 2tzx - txy \\ & + 4tyz + 2tsz - tys - 3txs - 2xz^2 s - 3y^2 s \\ & + yz^2 + zx + 2xyz + yz + 3zxs - 4ys - 2xs \\ & + 5yzs - 3xys + z^2 x - 2tz \end{aligned}$$

and the corresponding Dixon resultants yield the following triangularization

$$xy + y^2 - z^2 + xz - z + x + y$$

$$-(y^2 + 2y - z - yz - 2z^2 + 1)\, z(y + z + 1)$$

$$8(2z - 1)\, z(z - 1)(2z^2 + 3z - 2)$$

The solutions of the triangular system over the rationals are

$$(-r,r,0), (r,-3/2,1/2), (r,-2,1), (r,-1,0),$$
$$(0,1,1), (1/2,0,1/2), (3,-5,-2)$$

for any rational number r. These include the solutions of the original system

$$(0,0,0), (1/2,0,1/2), (1/2,-3/2,1/2),$$
$$(0,-2,1), (3,-5,-2)$$

One has to be careful with the application of this method in practice and should always check for the validity of the precondition. The following example illustrates this point.

Example 3 Compare the solutions of the following system with those of the Dixon triangularization described.

$$f = x^2 z - xz + xzy - yz$$

$$g = xzy + xz - xy - x$$

$$h = xz^2 y - 2xz^2 - yz^2 + 2z^2 + xzy - 2xz - yz + 2z$$

Solution: The rational solutions of this system are

$$(0,r,0),(0,0,-1),(r,-1,0),$$
$$(1,-1,r),(-2,2,1),(1,r,1)$$

for any rational numbers *r*. A triangularization as above yields the following triangular system

$$x^2z - xz + xzy - yz$$
$$yz^2(yz + z - y - 1)^2$$
$$6z^4(z^2 - 1)^2$$

with rational solutions

$$(0,0,-1),(1,0,-1),(r,s,0),$$
$$(r,-r,1),\ (1,-1,-1),(1,r,1)$$

for any rational numbers *r* and *s*. Note that the solution (1,-1,*r*) of the original system is not a solution of the triangularized one. In this case the method fails. The reason is, as one can easily verify, that each column of the Dixon matrix of the original system is a linear combination in the remaining ones. Thus, the precondition fails.

5 Conclusions

The Dixon triangularization described in the last subsection is in its preliminary stages. This is only a first step towards actually solving polynomial systems by this method. Some very basic questions currently under consideration are:

- What is the exact relation between the solution spaces of the original and the reduced system?
- What kind of assumptions are needed to ensure that the solution space of the original and the reduced system coincide?
- How do the Dixon extraneous factors affect the original solution?
- How can one tell whether the original system has finitely many solutions by examining the triangular system?
- How fast is this method compared with other methods?

Experiments indicate that this method is very fast compared with solutions found by using Groebner bases. It is also fast compared with other resultant methods. Unlike the one step elimination of other resultant methods we eliminate one variable at a time until triangularization.

There are also some disadvantages of the Dixon triangularization. The method may fail due to singularities. In this case reshuffling of polynomials may help. There are extra solutions that are not solutions of the original system. The method is not very efficient for high degree polynomial systems with many equations and variables. In those cases, however, all other basic methods also fail.

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