Delay-independent criterion of absolute stability for nonautonomous systems with variable delays

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Abstract: This paper deals with absolute stability conditions for multivariable systems with time-varying memoryless nonlinearities subject to sector conditions, and with variable delays. Assuming essentially a sector condition on the rate of variation of the nonlinearities and a bound on the derivative of the delays, we provide a stability criterion independent upon the size of the delays, expressed as a LMI condition. A numerical example of application is treated.

Key-words: delay systems, variable delays, absolute stability, delay-independent conditions, time-varying nonlinearities.

1 Introduction

We consider here a multivariable nonlinear control system given by the following delay differential equation:

$$\begin{cases} \dot{x} = Ax + \sum_{l=1}^{L} A_l x(t-h_l(t)) - B\psi(t,y), \\ y = Cx + \sum_{l=1}^{L} C_l x(t-h_l(t)),\ x_0^{(h)} = \phi\ , \end{cases} \tag{1}$$

where $n, p \in \mathbb{N} \setminus \{0\}$, $L \in \mathbb{N}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $A_l \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C_l \in \mathbb{R}^{p\times n}$, $0 < h_1(t) < \cdots < h_L(t) \leq h$ for any $t \in \mathbb{R}^+$, for a certain $h \in \mathbb{R}$, and $x_t^{(h)}(s) = x(t+s)$ for $s \in [-h, 0]$. For constant delays, sufficient conditions for absolute stability of system (1) are provided in [12, 11] for stationnary nonlinearities, and in [2] for nonstationary nonlinearities. Also, a generalization of circle criterion is provided in [5] for systems with variable delays. The previous results are *delay-dependent*, in the sense that their hypotheses take into account explicitly the magnitude of the delays.

On the other hand, when the latter are unknown, one may be interested [8] by *delay-independent* conditions, more precisely by results *independent upon the size of the delay.* Delay-independent absolute stability results have been given in [6, 14], in the case where the nonlinearity is stationnary with undelayed input ($y = Cx(t)$ in (1)). A generalization to general inputs and nonstationary nonlinearities is given in [3].

In the present paper, the previous results are now generalized, in order to apply to systems with variable delays. As in [3], the conditions are expressed in terms of Linear Matrix Inequalities [4]. Also, the nonstationarity of the nonlinearities is taken into account by assuming a generalized sector inequality on the derivative of the nonlinearity, an idea borrowed from [13, 7] (for rational systems) and expressed under a slightly weaker form in [1]. The main hypothesis on the delays is that their variation wrt time is bounded.

For the sake of simplicity, the results are given only for a system with two delays, namely (2).

Extending the method to general case (1) only necessitates some (cumbersome) computations.

$$\begin{cases} \dot{x} = Ax + A_\alpha x(t - h_\alpha(t)) - B\psi(t, y), \\ y = Cx + C_\beta x(t - h_\beta(t)) \ . \end{cases} \tag{2}$$

In Section 2 are formulated criteria without restrictions on the variations of ψ wrt time (Theorems 1 to 3). Stronger criteria are given in Section 3 (Theorems 4 to 6). In Theorems 1 and 4, h_α and h_β are independent, sharper results are provided in the case where $h_\alpha = h_\beta$ (system (3), Theorems 2 and 5) or $h_\alpha = 2h_\beta$ (system (4), Theorems 3 and 6). Finally, an example illustrating the method is provided in Section 4.

$$\begin{cases} \dot{x} &= Ax + A_\alpha x(t - h(t)) - B\psi(t, y), \\ y &= Cx + C_\beta x(t - h(t)) \ . \end{cases} \tag{3}$$

$$\begin{cases} \dot{x} &= Ax + A_\alpha x(t - 2h(t)) - B\psi(t, y), \\ y &= Cx + C_\beta x(t - h(t)) \ . \end{cases} \tag{4}$$

We do not consider here the issues of existence and uniqueness of the solutions, as they have been extensively studied: in the sequel, we only assume the existence of *global solutions of* (1), i.e. $\forall \phi \in \mathcal{C}([-h, 0]; \mathbb{R}^n)$, $\exists x \in \mathcal{C}([-h, +\infty))$, x absolutely continuous (AC) on $[0, +\infty)$, s.t. $x_0^{(h)} = \phi$ and (1) holds almost everywhere (a.e.) on $[0, +\infty)$. The results given below concern the asymptotic behavior of these global solutions.

Notations For the sake of simplicity, one writes $x, x_\alpha, x_\beta, x_{\alpha\beta}, x_{\beta\beta}, y_\beta, \psi, \psi_\beta$ instead of $x(t), x(t - h_\alpha(t)), x(t - h_\beta(t)), x(t - h_\alpha(t) - h_\beta(t)), x(t - 2h_\beta(t)), y(t - h_\beta(t)), \psi(t, y(t)), \psi(t - h_\beta(t), y(t - h_\beta(t)))$. Also, I_n denotes the $n \times n$ identity matrix, $0_{n\times p}$ a null $n \times p$ matrix (0_n when $p = n$). Last, for M_k square matrices, $k = \overline{1, K}$, $\mathrm{diag}\{M_1, \ldots, M_K\}$ is defined recursively by

$$\begin{pmatrix} M_1 & 0 \\ 0 & \mathrm{diag}\{M_2, \ldots, M_K\} \end{pmatrix} \ .$$

2 Delay-independent LMI criterion

Theorem 1 (General case). *Assume that the following Hypotheses hold.*

(H1) *The nonlinearity ψ is measurable, decentralized [9] (that is: $\forall i = \overline{1, p}$, $\psi_i(t, y) = \psi_i(t, y_i)$), and there exists a diagonal matrix $K \geq 0$ such that, $\forall (t, y) \in \mathbb{R}^+ \times \mathbb{R}^p$,*

$$\psi(t, y)^T (\psi(t, y) - Ky) \leq 0 \ .$$

(H2) *The functions h_α, h_β are AC, and there exists $\delta \in (0, 1]$ such that*

$$\dot{h}_\alpha, \dot{h}_\beta \leq 1 - \delta \quad t - a.e.$$

Assume that LMI (5,6) is feasible (where the symmetric matrices $P, Q_\alpha, Q_\beta \in \mathbb{R}^{n\times n}$ are the variables). Then the origin of system (2) is uniformly globally asymptotically stable.

$$P > 0, \quad Q_\alpha, Q_\beta \geq 0, \quad R < 0 \ . \tag{5}$$

Sketch of the proof. Consider the following Liapunov function candidate

$$V(t, \phi) \stackrel{\mathrm{def}}{=} \phi(0)^T P\phi(0) + \int_{-h_\alpha(t)}^0 \phi(s)^T Q_\alpha \phi(s) \, ds + \int_{-h_\beta(t)}^0 \phi(s)^T Q_\beta \phi(s) \, ds \ .$$

To prove Theorem 1, differentiate $V(t, x_t^{(h)})$ wrt time and, adding the term $-2\psi^T(\psi - Ky)$ (which is nonnegative, due to sector condition (H1)), write that $\dot{V} \leq X(t)^T RX(t)$ $t-$a.e., where $X(t) \stackrel{\mathrm{def}}{=} (x^T \ \psi^T \ x_\alpha^T \ x_\beta^T)^T$. Now, the hypotheses on the regularity of the global solutions and of the delays imply that the map $t \mapsto V(t)$ is AC, so $\exists c > 0$ "small", such that $V(t) + c\int_0^t \|x(s)\|^2 \, ds = V(0) + \int_0^t \frac{dV}{ds} \, ds + c\int_0^t \|x(s)\|^2 \, ds$ is a nonincreasing function of t, as its derivative is a.e. nonpositive. On the other hand, $\exists c' > 0$ such that, $\forall (t, \phi) \in \mathbb{R}^+ \times \mathcal{C}([-h, 0])$, $c'\|\phi(0)\| \leq V(t, \phi) \leq \frac{1}{c'}\|\phi\|_{\mathcal{C}([-h,0])}$. One then achieves the proof as in [10, Theorem 31.1]. ♠

It is also possible to get local stability results, see [2, 3]. Results may be extended without difficulty when (H2) is replaced by: $\dot{h}_\alpha \leq 1 - \delta_\alpha$, $\dot{h}_\beta \leq 1 - \delta_\beta$.

We provide now two sharper results, in the case where $h_\alpha = h_\beta$ (in which case $x_\alpha = x_\beta$) or $h_\alpha = 2h_\beta$ (and then $\dot{h}_\beta = \frac{\dot{h}_\alpha}{2} \leq \frac{1-\delta}{2}$).

$$R \stackrel{\text{def}}{=} \begin{pmatrix} A^TP + PA + Q_\alpha + Q_\beta & C^TK - PB & PA_\alpha & 0_n \\ KC - B^TP & -2I_p & 0_{p\times n} & KC_\beta \\ A_\alpha^T P & 0_{n\times p} & -\delta Q_\alpha & 0_n \\ 0_n & C_\beta^T K & 0_n & -\delta Q_\beta \end{pmatrix} . \quad (6)$$

$$R_2 \stackrel{\text{def}}{=} \begin{pmatrix} A^TP + PA + Q_\alpha + Q_\beta & C^TK - PB & PA_\alpha & 0_n \\ KC - B^TP & -2I_p & 0_{p\times n} & KC_\beta \\ A_\alpha^T P & 0_{n\times p} & -\delta Q_\alpha & 0_n \\ 0_n & C_\beta^T K & 0_n & -\frac{1+\delta}{2} Q_\beta \end{pmatrix} . \quad (7)$$

Theorem 2 (Case $h_\alpha = h_\beta$). *Assume that Hypotheses* (H1), (H2) *hold and that LMI* (8,9) *is feasible. Then the origin of system* (3) *is uniformly globally asymptotically stable.*

$$P > 0, \quad Q \geq 0, \quad R_1 < 0 . \quad (8)$$

$$R_1 \stackrel{\text{def}}{=} \begin{pmatrix} A^TP + PA + Q & C^TK - PB & PA_\alpha \\ KC - B^TP & -2I_p & KC_\beta \\ A_\alpha^T P & C_\beta^T K & -\delta Q \end{pmatrix} . \quad (9)$$

Theorem 3 (Case $h_\alpha = 2h_\beta$). *Assume that Hypotheses* (H1), (H2) *hold and that LMI* (7,10) *is feasible. Then the origin of system* (4) *is uniformly globally asymptotically stable.*

$$P > 0, \quad Q_\alpha, Q_\beta \geq 0, \quad R_2 < 0 . \quad (10)$$

3 Delay-independent LMI criterion for nonlinearities with restricted time-variations

The results of Theorems 1 to 3 are in a sense analog to *circle criterion*: they use as Lyapunov function a quadratic function of the state $x_t^{(h)}$, and are valid for systems with nonstationary nonlinearities as well. In order to refine the analysis, one may try to mimick the argument leading to *Popov criterion* and add to V a Lur'e term, namely $2\sum_{i=1}^p \eta_i K_i \int_0^{y_i(t)} \psi_i(t,z)\ dz$.

However, when the input of the nonlinearity is delayed ($C_\beta \neq 0$), this operation introduces in $\dot V$ terms in $x_{\alpha\beta}$ and $x_{\beta\beta}$. To counterbalance them, one is led to introduce new terms in V, in order now to bound $\dot V$ by a negative definite quadratic form in $x, \psi, x_\alpha, x_\beta, \psi_\beta, x_{\alpha\beta}, x_{\beta\beta}$. The following result is proved by considering the evolution of $V_\beta(t, x_t^{(2h)})$, where the Lyapunov function V_β is defined (for $t \geq h$) by

$$\begin{aligned} V_\beta(t,\phi) \stackrel{\text{def}}{=} & V(t,\phi) + \phi(-h_\beta(t))^T P_\beta \phi(-h_\beta(t)) \\ & + \int_{-h_\alpha(t)-h_\beta(t)}^0 \phi(s)^T Q_{\alpha\beta} \phi(s)\ ds \\ & + \int_{-2h_\beta(t)}^0 \phi(s)^T Q_{\beta\beta} \phi(s)\ ds \\ & +2\sum_{i=1}^p \eta_i K_i \int_0^{(C\phi)_i(0)} \psi_i(t,z)\ dz . \end{aligned}$$

Theorem 4 (General case). *Assume that Hypotheses* (H1), (H2) *hold, together with*

(H0) *For any $y \in \mathbb{R}^p$, $t \mapsto \psi(t,y)$ is locally Lipschitz (and hence t-a.e. differentiable), with a Lipschitz constant locally integrable wrt y.*

Assume that there exists diagonal matrices D_j, $j = \overline{1,3}$ such that LMI (6,12,15) *is feasible and that the following Hypothesis holds (with the same η and D_j)*

(H3) *For almost any $t \in \mathbb{R}^+$, $\forall y \in \mathbb{R}^p$, $\forall i = \overline{1,p}$,*

$$\eta_i \left(\int_0^{y_i} \frac{\partial \psi_i}{\partial t}(t,z)\ dz - D_{1,i} y_i^2 - D_{2,i} y_i \psi_i(t,y_i) - D_{3,i}\psi_i(t,y_i)^2 \right) \leq 0 . \quad (11)$$

Then the origin of system (2) *is uniformly globally asymptotically stable.*

$$P > 0, \quad P_\beta, Q_\alpha, Q_\beta, Q_{\alpha\beta}, Q_{\beta\beta} \geq 0, \quad \eta = \text{diag}\{\eta_i\}, \zeta = \text{diag}\{\zeta_i\} \geq 0 \ , \quad R_\beta < 0 \ . \tag{12}$$

$$P > 0, \quad P_\beta, Q, Q_\beta \geq 0, \quad \eta = \text{diag}\{\eta_i\}, \zeta = \text{diag}\{\zeta_i\} \geq 0 \ , \quad R_{\beta,1} < 0 \ . \tag{13}$$

$$P > 0, \quad P_\beta, Q_\alpha, Q_\beta, Q_{\alpha\beta}, Q_{\beta\beta} \geq 0, \quad \eta = \text{diag}\{\eta_i\}, \zeta = \text{diag}\{\zeta_i\} \geq 0 \ , \quad R_{\beta,2} < 0 \ . \tag{14}$$

Sketch of proof. First, verify that the map $t \mapsto V_\beta(t, x_t^{(2h)})$ is AC, essentially because $\forall t, t' \in \mathbb{R}^+$, $\forall i = \overline{1,p}$, $\left|\int_0^{y_i(t)} \psi_i(t,z)\ dz - \int_0^{y_i(t')} \psi_i(t',z)\ dz\right| \leq |t - t'| \int_0^{y_i(t)} \lambda_i(z)\ dz + K_i \max\{|y_i(t)|, |y_i(t')|\}\ |y_i(t) - y_i(t')|$, where λ_i is the Lispchitz constant of ψ, defined by Hypothesis (H0). Also, as $-1 + \dot{h}_\alpha + \dot{h}_\beta, -1 + 2\dot{h}_\beta \leq 1 - 2\delta$, one gets, by addition of $-2\psi_\beta^T \zeta(\psi_\beta - Ky_\beta) \geq 0$, that $\dot{V}_\beta \leq X_\beta(t)^T R_\beta X_\beta(t)$ $t-$a.e., where $X_\beta(t) \stackrel{\text{def}}{=} (x^T\ \psi^T\ x_\alpha^T\ x_\beta^T\ \psi_\beta^T\ x_{\alpha\beta}^T\ x_{\beta\beta}^T)^T$. Estimates as in the proof of Theorem 1 (but now in space $\mathcal{C}([-2h, 0])$) lead to the conclusion, again by [10, Theorem 31.1]. ♠

Theorem 1 appears as a subcase of Theorem 4. In the case where the delays are constant and do not appear in the input nonlinearity ($C_\beta = 0$), the results given in [6, 14] are found.

Hypothesis (H3) is a *generalized sector condition*, fulfilled e.g. when there exists a measurable map Δ with diagonal matrix values, s.t. $t-$a.e. in $\mathbb{R}^+$, $\forall y \in \mathbb{R}^p$, $y^T \eta \left(\frac{\partial \psi}{\partial t}(t,y) - \Delta(t) y\right) \leq 0$, and the matrices D_j are then given by $D_1 = \frac{1}{2} \text{ess sup}\{\Delta(t)\ :\ t \geq 0\}$, $D_2 = D_3 = 0$. Multiplication by η_i in (11) indicates that this constraint is inactive when $\eta_i = 0$. If such a formula is fulfilled for $\eta \not\geq 0$, consider the nonlinearity $\text{sgn}\,\eta\ \psi(t,y) + \frac{1}{2}(1 - \text{sgn}\,\eta) Ky$ instead of $\psi(t,y)$, see [2, 3].

Sharper results are given in the sequel, in the case where $h_\alpha = h_\beta$ (and then $x_\alpha = x_\beta$, $x_{\alpha\beta} = x_{\beta\beta}$) or $h_\alpha = 2h_\beta$ ($x_\alpha = x_{\beta\beta}$, $1 - \dot{h}_\alpha - \dot{h}_\beta \leq \frac{1-3\delta}{2}$).

Theorem 5 (Case $h_\alpha = h_\beta$). *Assume that Hypotheses* (H0), (H1), (H2) *hold, that there exists diagonal matrices D_j, $j = \overline{1,3}$ such that LMI* (9,13,16) *is feasible, and that Hypothesis* (H3) *holds (with the same η and D_j). Then the origin of system* (3) *is uniformly globally asymptotically stable.*

Theorem 6 (Case $h_\alpha = 2h_\beta$). *Assume that Hypotheses* (H0), (H1), (H2) *hold, that there exists diagonal matrices D_j, $j = \overline{1,3}$ such that LMI* (7,14,17) *is feasible, and that Hypothesis* (H3) *holds (with the same η and D_j). Then the origin of system* (4) *is uniformly globally asymptotically stable.*

4 Illustrative numerical example

As an illustration, let us consider the following scalar equation

$$\dot{x} = -x + 0.9 x_\alpha - \psi(t, x + 0.1 x_\beta) \ , \tag{18}$$

which may be written under the form (2) with

$$A = -1,\ A_\alpha = 0.9,\ B = 1,\ C = 1,\ C_\beta = 0.1 \ ,$$

and suppose that $D_2 = D_3 = 0$. All computations to be presented have been achieved using the Scilab package LMITOOL[1].

For given values of D_1 and δ, the maximal value of K allowed for using the Theorems is shown in Tables 1 to 3. This defines a *robustness margin*, which is larger when δ is closer to 1 or when D_1 is closer to 0, as could be foreseen. Also, linking h_α and h_β leads to larger margins. Remark that $\delta = 1.00$ e.g. when $\dot{h}_\alpha = \dot{h}_\beta = 0$, and $D_1 = 0$ e.g. when $\partial\psi/\partial t \equiv 0$.

[1] Scilab is a free software developed by INRIA, which is distributed with all its source code. For the distribution and details, see Scilab's homepage on the web at the address `http://www-rocq.inria.fr/scilab/`

$D_1 \backslash \delta$	1.00	0.95	0.90
0 (Th. 4)	225	158	96.0
1 (Th. 4)	103	37.1	18.4
$+\infty$ (Th. 1)	39.9	29.1	18.4

Table 1: Maximal value of K permitted for proving absolute stability of (18) – General case.

$D_1 \backslash \delta$	1.00	0.95	0.90
0 (Th. 5)	713	602	488
1 (Th. 5)	695	584	468
$+\infty$ (Th. 2)	305	280	254

Table 2: Maximal value of K permitted for proving absolute stability of (18) – Case $h_\alpha = h_\beta$.

$D_1 \backslash \delta$	1.00	0.95	0.90
0 (Th. 6)	276	202	128
1 (Th. 6)	106	38.9	19.5
$+\infty$ (Th. 3)	39.9	29.8	19.5

Table 3: Maximal value of K permitted for proving absolute stability of (18) – Case $h_\alpha = 2h_\beta$.

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\begin{aligned}
R_\beta \overset{\text{def}}{=}\; & \operatorname{diag}\{R, 0_{2n+p}\} + \operatorname{diag}\left\{ \begin{pmatrix} 2C^T\eta D_1 KC & C^T\eta D_2 K & 0_n & 2C^T\eta D_1 KC_\beta \\ \eta D_2 KC & 2\eta D_3 K & 0_{p\times n} & \eta D_2 KC_\beta \\ 0_n & 0_{n\times p} & 0_n & 0_n \\ 2C_\beta^T\eta D_1 KC & C_\beta^T \eta D_2 K & 0_n & 2C_\beta^T \eta D_1 KC_\beta \end{pmatrix}, 0_{2n+p} \right\} \\
& + \operatorname{diag}\left\{ Q_{\alpha\beta} + Q_{\beta\beta}, 0_{n+p}, \begin{pmatrix} A^T P_\beta + P_\beta A & C^T K\zeta - P_\beta B & P_\beta A_\alpha & 0_n \\ \zeta KC - B^T P_\beta & -2\zeta & 0_{p\times n} & \zeta KC_\beta \\ A_\alpha^T P_\beta & 0_{n\times p} & (1-2\delta)Q_{\alpha\beta} & 0_n \\ 0_n & C_\beta^T K\zeta & 0_n & (1-2\delta)Q_{\beta\beta} \end{pmatrix} \right\} \\
& + \operatorname{diag}\left\{ \begin{pmatrix} 0_n & A^T C^T K\eta & 0_n & 0_n & 0_{n\times p} & 0_n \\ \eta KCA & -2\eta KCB & \eta KCA_\alpha & \eta KC_\beta A & -\eta KC_\beta B & \eta KC_\beta A_\alpha \\ 0_n & A_\alpha^T C^T K\eta & 0_n & 0_n & 0_{n\times p} & 0_n \\ 0_n & A^T C_\beta^T K\eta & 0_n & 0_n & 0_{n\times p} & 0_n \\ 0_{p\times n} & -B^T C_\beta^T K\eta & 0_{p\times n} & 0_{p\times n} & 0_p & 0_{p\times n} \\ 0_n & A_\alpha^T C_\beta^T K\eta & 0_n & 0_n & 0_{n\times p} & 0_n \end{pmatrix}, 0_n \right\}. \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
R_{\beta,1} \overset{\text{def}}{=}\; & \operatorname{diag}\{R_1, 0_{n+p}\} + \operatorname{diag}\left\{ Q_\beta, 0_p, \begin{pmatrix} A^T P_\beta + P_\beta A & C^T K\zeta - P_\beta B & P_\beta A_\alpha \\ \zeta KC - B^T P_\beta & -2\zeta & \zeta KC_\beta \\ A_\alpha^T P_\beta & C_\beta^T K\zeta & (1-2\delta)Q_\beta \end{pmatrix} \right\} \\
& + \begin{pmatrix} 0_n & A^T C^T K\eta & 0_n & 0_{n\times p} & 0_n \\ \eta KCA & -2\eta KCB & \eta K(CA_\alpha + C_\beta A) & -\eta KC_\beta B & \eta KC_\beta A_\alpha \\ 0_n & (A_\alpha^T C^T + A^T C_\beta^T)K\eta & 0_n & 0_{n\times p} & 0_n \\ 0_{p\times n} & -B^T C_\beta^T K\eta & 0_{p\times n} & 0_p & 0_{p\times n} \\ 0_n & A_\alpha^T C_\beta^T K\eta & 0_n & 0_{n\times p} & 0_n \end{pmatrix} \\
& + \operatorname{diag}\left\{ \begin{pmatrix} 2C^T\eta D_1 KC & C^T\eta D_2 K & 2C^T\eta D_1 KC_\beta \\ \eta D_2 KC & 2\eta D_3 K & \eta D_2 KC_\beta \\ 2C_\beta^T \eta D_1 KC & C_\beta^T \eta D_2 K & 2C_\beta^T \eta D_1 KC_\beta \end{pmatrix}, 0_{n+p} \right\}. \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
R_{\beta,2} \overset{\text{def}}{=}\; & \operatorname{diag}\{R_2, 0_{n+p}\} + \operatorname{diag}\left\{ \begin{pmatrix} 2C^T\eta D_1 KC & C^T\eta D_2 K & 0_n & 2C^T\eta D_1 KC_\beta \\ \eta D_2 KC & 2\eta D_3 K & 0_{p\times n} & \eta D_2 KC_\beta \\ 0_n & 0_{n\times p} & 0_n & 0_n \\ 2C_\beta^T\eta D_1 KC & C_\beta^T \eta D_2 K & 0_n & 2C_\beta^T \eta D_1 KC_\beta \end{pmatrix}, 0_{n+p} \right\} \\
& + \operatorname{diag}\left\{ Q_{\alpha\beta} + Q_{\beta\beta}, 0_p, \begin{pmatrix} -\delta Q_{\beta\beta} & 0_n & C_\beta^T K\zeta & 0_n \\ 0_n & A^T P_\beta + P_\beta A & C^T K\zeta - P_\beta B & P_\beta A_\alpha \\ \zeta KC_\beta & \zeta KC - B^T P_\beta & -2\zeta & 0_{p\times n} \\ 0_n & A_\alpha^T P_\beta & 0_{n\times p} & \frac{1-3\delta}{2} Q_{\alpha\beta} \end{pmatrix} \right\} \\
& + \begin{pmatrix} 0_n & A^T C^T K\eta & 0_n & 0_n & 0_{n\times p} & 0_n \\ \eta KCA & -2\eta KCB & \eta KCA_\alpha & \eta KC_\beta A & -\eta KC_\beta B & \eta KC_\beta A_\alpha \\ 0_n & A_\alpha^T C^T K\eta & 0_n & 0_n & 0_{n\times p} & 0_n \\ 0_n & A^T C_\beta^T K\eta & 0_n & 0_n & 0_{n\times p} & 0_n \\ 0_{p\times n} & -B^T C_\beta^T K\eta & 0_{p\times n} & 0_{p\times n} & 0_p & 0_{p\times n} \\ 0_n & A_\alpha^T C_\beta^T K\eta & 0_n & 0_n & 0_{n\times p} & 0_n \end{pmatrix}. \qquad (17)
\end{aligned}
$$