Periodic Disturbance Rejection: Part I – Minimum Variance Control[‡]

T. O' MAHONY & C.J. DOWNING

Department of Electronics and Electrical Eng.
Cork Institute of Technology,
Bishopstown, Cork
IRELAND

Abstract:- The minimum variance controller is examined in terms of its ability to reject ramp and periodic disturbances and a design procedure developed to enable this controller to reject such disturbances. In parallel the PID controller is also examined and shown to be deficient in this regard. A novel extension to the PID controller equips it with the ability to reject both ramp and periodic disturbances. The problem of tuning this controller is then addressed and it is shown that the reinterpretation of the PID structure as a minimum variance controller leads to a systematic tuning procedure. This tuning procedure was then applied to the modified PID controller and shown to yield suitable results when the system was perturbed by step or ramp functions. It is then shown that the method fails when a periodic disturbance is applied and the reason for this is highlighted. In the companion paper by the same authors an alternative tuning method is presented which is applicable for this situation.

Key-words:- Minimum Variance Control, PID Control, Deterministic Disturbance Rejection. *Proc.pp..1891-1896*

1. Introduction

The presence of disturbances is one of the main reasons for using control. Without disturbances there is no justification for using feedback and consequently much literature is devoted to the subject of disturbance rejection. The most common control loop disturbances typically act upon the load and may be classified as either periodic or aperiodic. Impulse, step, ramp and random functions comprise the latter category, while the former is generally described in terms of a sinusoidal function. While each of these disturbance models places its own particular demand upon the control loop, textbooks tend to restrict disturbance response analysis to the effect of the step/offset disturbance upon the steady state response. In this case, to ensure zero steady state error it is well known that the controller must contain integral action. Many of the early controllers e.g. the minimum variance controller of Åström and Wittenmark [1] and the pole-placement controller of Wellstead and his co-workers [8], did not have inherent integral action and various *ad-hoc* methods were employed to ensure zero steady state offset. Tuffs and Clarke [7] remedied the situation when they presented a unified approach to the offset problem based on the CARIMA model rather than the traditional CARMA model. This paper utilises a similar idea to derive a minimum variance controller that can reject the various types of disturbances previously described. This is then contrasted with the PID controller which due to its limited order is unable to reject ramp- or sinusoidal-like disturbances. The traditional three term controller is then extended to encompass both ramp and sinusoidal disturbance rejection. Since the tuning of this controller may be problematic a systematic design procedure based on a minimum variance objective function is developed. This tuning technique may be applied to the modified PID controller to yield a closed-loop which has a MV response to set-point and is capable of rejecting both step and ramp disturbances. The tuning technique is subsequently shown to yield poor results when applied to a process, controlled by the modified PID controller and disturbed by a periodic function. The

[‡] This research was funded by the CIT Scholarship Scheme. The authors wish to gratefully acknowledge this support

companion paper, [6], redresses this problem by developing an alternative tuning technique.

2. MV Controller

The Minimum Variance (MV) controller is the stochastic equivalent of the One Step Ahead Controller, and is primarily concerned with the minimisation of the output variance at the sampling instants. In the derivation of the MV controller a process model of the form:

$$A(z^{-1})y(t) = z^{-k}B(z^{-1})u(t) + C(z^{-1})x(t) \qquad (1)$$

where $A(z^{-1})$, $C(z^{-1})$ are monic FIR polynomials in the backward shift operator z^{-1} of order *na* and *nc* respectively, while $B(z^{-1})$ is a FIR polynomial of order *nb*. In equation 1 *y(t)* is the process output, *u(t)* is the control signal, *x* is a white noise source and *k* is the process dead-time or time-delay. The development of the MV controller centres on a Diophantine equation i.e.

$$C(z^{-1}) = E(z^{-1})A(z^{-1}) + z^{-k}F(z^{-1}) \qquad (2)$$

which yields a minimum degree solution if $ne=k-1$ and $nf=max(na-1,nc-1)$. The solution to this identity is used to define the *k*-step ahead predictor:

$$\hat{y}(t+k\,|\,t) = Fy(t) + Gu(t) \qquad (3)$$

where $G(z^{-1})=E(z^{-1})B(z^{-1})$. Choosing an objective function which minimises fluctuations between the set-point, *r(t)*, and predicted output i.e.

$$J = E\left\{\left[\hat{y}(t+k\,|\,t) - r(t)\right]^2\right\} \qquad (4)$$

and minimising (4) w.r.t. *u(t)* yields the following MV control law

$$u(t) = \frac{r(t) - Fy(t)}{G} \qquad (5)$$

This control law clearly cancels the process numerator and therefore cannot be used to control non-minimum phase processes. Gawthrop [3] and Clarke and Gawthrop [2] present a Generalised Minimum Variance controller that avoids this problem.

The closed-loop response of the MV controller to set-point, *r(t)*, disturbances at the process output, *d(t)*, and disturbances occurring at the process input, $d_1(t)$, is given by:

$$y(t) = \frac{z^{-k}B}{GA + z^{-k}BF}r(t) + \frac{z^{-k}BG}{GA + z^{-k}BF}d_1(t) + \frac{AG}{GA + z^{-k}BF}d(t) \qquad (6)$$

This equation clearly indicates that for disturbance rejection to be possible the polynomial $G(z^{-1})$ must contain a model of the disturbance. In general, the solution of eqn 2 will not yield such a desirable *G* polynomial and consequently the basic MV controller does not reject periodic or aperiodic disturbances; nor is it feasible to simply augment the block diagram by convoluting *G* with say $D=(1-z^{-1})$ as the arbitrary insertion of an integrator in the forward path will reduce stability margins and may destabilise the controller. A more systematic approach is to augment the Diophantine equation 2 such that it explicitly includes the poles of the disturbance and then to re-derive the *F* and *G* polynomials on the basis of this equation. The modified Diophantine equation is given by:

$$C(z^{-1}) = E(z^{-1})D(z^{-1})A(z^{-1}) + z^{-k}F(z^{-1}) \qquad (7)$$

The polynomial $D(z^{-1})$ models the disturbance and typically consists of one of the following:

- Step - $D(z^{-1}) = (1 - z^{-1})$
- Ramp - $D(z^{-1}) = (1 - z^{-1})^2$ (8)
- Sinusoid - $D(z^{-1}) = (1 - z^{-1})(1 - 2\cos(wt_s)z^{-1} + z^{-2})$

In the latter case *w* is the frequency of the disturbance and t_s is the sampling period. The inclusion of the integrator is not strictly necessary to reject the sinusoid but is essential if offset free control is also a requirement. A minimum degree solution to equation 6 is obtained by choosing $ne = k-1$, and $nf=max(na+nd-1,nc-1)$. This yields a predictor as defined by equation 3 with

$$G(z^{-1}) = E(z^{-1})B(z^{-1})D(z^{-1}) \qquad (9)$$

Minimising the cost of equation 4 yields a controller as defined by equation 5. This controller may now be used in a loop that is disturbed by both stochastic and general deterministic disturbances.

3. PID Control

Thus far it has been illustrated how minimum variance controller may be augmented to successfully reject ramp and periodic disturbances. However, since it is estimated that PID controllers control some 90% of all control loops it is important to analyse the capabilities of this popular controller. This analysis will be strictly limited to the digital domain though the ideas are equally applicable to the Laplace model of the PID controller. Throughout the discussion it is assumed that the PID controller is in the following form:

$$u(t) = \left[K_p + \frac{K_i}{(1 - z^{-1})} + K_d(1 - z^{-1})\right]e(t) \qquad (10)$$

$$= \left[\frac{(K_p + K_i + K_d) - (K_p + 2K_d)z^{-1} + K_d z^{-2}}{1 - z^{-1}} \right] \mathrm{e(t)}$$

$$= \frac{S(z^{-1})}{R(z^{-1})} \mathrm{e(t)}$$

If a controller as defined above is used in conjunction with a generic process model as defined in equation 1, the closed-loop will then be defined by

$$y(t) = \frac{z^{-k}SB}{AR + z^{-k}SB} r(t) + \frac{z^{-k}BR}{AR + z^{-k}SB} d_1(t) + \frac{AR}{AR + z^{-k}SB} d(t) \qquad (11)$$

Again it is clear that if deterministic disturbances are to be successfully rejected, the controller denominator will have to include the poles of the disturbance. In the PID case the controller denominator is strictly limited to including an integrator i.e. $R(z^{-1})=(1-z^{-1})$, and consequently the first conclusion is that the PID controller does not have the ability to track ramp-like inputs or to reject ramp-like disturbances without offset. In addition it does not have the ability to completely reject periodic disturbances.

To incorporate deterministic disturbance rejection it is necessary to modify the PID controller. An intuitive, and simple, modification is to include another gain term that acts on the disturbance i.e.

$$u(t) = \left[K_p + \frac{K_i}{(1 - z^{-1})} + \frac{K_e}{D'(z^{-1})} + K_d(1 - z^{-1}) \right] \mathrm{e(t)} \qquad (12)$$

Since this controller already incorporates an integrator the $D'(z^{-1})$ term is a limited version of equation set 8 and may consist of the following

- Ramp $D'(z^{-1}) = (1 - z^{-1})^2$
- Sinusoid $D'(z^{-1}) = (1 - 2\cos(wt_s)z^{-1} + z^{-2})$

Two obvious questions now arise

1) Is it possible to find coefficients for this controller that will yield a stable closed-loop?
2) How are these coefficients to be found?

The second question will be addressed first and in so doing the answer to the first will emerge.

With a standard PID controller there exists a number of possibilities for determining its parameters. These may be categorised as follows

- The use of iterative tuning methods
- The use of tuning rules
- Minimising a performance criterion
- Placing the closed-loop poles

For a detailed review of these various design methodologies see O' Dwyer [4]. The first of these design procedures typically incorporates manual 'trial and error' tuning and the use of graphical techniques in either the time or the frequency domain. Manual tuning can easily be applied to the modified controller of equation 12 though it is expected that the selection of appropriate controller gains will be complicated by the inclusion of the disturbance dynamics.

The second, and perhaps most widely used, possibility for tuning a PID controller is to apply one of the abundance of existing tuning rules to the process and hope that it yields suitable results. This methodology could also be applied to a process controlled by equation 12, though, since the tuning rules were developed for the standard three term controller, it would be optimistic in the extreme to expect them to yield satisfactory results. In addition no tuning rules are available (to the authors' knowledge) which are applicable to the modified controller. Even if such methods were available it will be shown that there are strict limitations on the possible locations of the closed-loop poles when equation 12 is implemented and consequently any *ad-hoc* rules are likely to be unsuccessful.

Tuning the PID controller through the placement of the closed-loop poles is a very powerful design technique as it allows a wide range of controller objectives to be specified. The application of this tuning method is deferred to the second part of this paper where it is discussed in some detail by the authors.

The three-term controller may also be tuned by minimising a performance (or optimisation) criterion. Simple optimisation criteria include the integral of the absolute value of the error (IAE) criterion and the integral of time multiplied by the absolute value of the error (ITAE) criterion. Here a quadratic performance criterion, identical to equation 4, will be minimised and hence the PID controller will have a MV interpretation. To illustrate this design methodology it is assumed that a controller structure, designed to reject sinusoidal and offset disturbances, as defined by equation 13 is desirable.

$$u(t) = \left[K_p + \frac{K_i}{(1 - z^{-1})} + \frac{K_e}{\left(1 - 2\cos(wt_s)z^{-1} + z^{-2}\right)} + K_d(1 - z^{-1}) \right] \mathrm{e(t)} \qquad (13)$$

Since this modified PID controller still retains a strict structure there are limitations imposed on the maximum order of the process model that can be used. These restrictions are that:

$$A(z^{-1}) = 1 + a_1 z^{-1} + a_2 z^{-2}$$
$$B(z^{-1}) = b_0 \qquad (14)$$
$$k=1$$

Equation 13 can equivalently be represented in difference equation form as,

$$u(t) = \frac{s_0 + s_1 z^{-1} + s_2 z^{-2} + s_3 z^{-3} + s_4 z^{-4}}{1 + r_1 z^{-1} + r_2 z^{-2} + r_3 z^{-3}} e(t) \qquad (15)$$

where

$$s_0 = K_p + K_i + K_e + K_d$$
$$s_1 = -K_p(1+2b) - 2bK_i - K_e - K_d(2+2b)$$
$$s_2 = K_p(1+2b) + K_i + 2K_p(1+2b)$$
$$s_3 = -K_p - K_d(2+2b)$$
$$s_4 = K_d \qquad (16)$$
$$r_1 = -(1+2\cos(wt_s))$$
$$r_2 = 1+2\cos(wt_s)$$
$$r_3 = -1 \qquad (17)$$

and $b = cos(wt_s)$. By comparing controller structures it is evident that a MV interpretation is possible if

$$R(z^{-1}) = D(z^{-1});\ S(z^{-1}) = F(z^{-1})/b_0; \qquad (18)$$

Since the disturbance model specifies the controller denominator, the controller may be implemented by solving equation 7 to obtain F and subsequently using equation 18 to obtain S.

In many cases the justification for using a PID controller is to present to the operator the freedom to 'tweak' the controller gains and modify the closed-loop response. This implies that in practice equation 13 (or a variant thereof) will be implemented on a digital computer, and consequently it will be necessary to compute the controller gains (K_p, K_i, K_e and K_d) given the polynomial $S(z^{-1})$, or *vice-versa*, to determine $S(z^{-1})$ given the controller gains. To determine the controller gains from the $S(z^{-1})$ polynomial the system 19 will have to be solvable.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -(1+2b) & -2b & -1 & -2(1+b) \\ 1+2b & 1 & 0 & 2(1+2b) \\ -1 & 0 & 0 & -2(1+b) \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} K_p \\ K_i \\ K_e \\ K_d \end{bmatrix} = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \qquad (19)$$

where $b = cos(wt_s)$. Equation 19 represents an over-determined system of linear equations and an exact analytical solution is only possible if

$$s_0 + s_1 = -2K_p b + K_i(1 - 2b) - K_d(1 + 2b) \qquad (20)$$

where $b = cos(wt_s)$. Further insight into equation 20 may be obtained by examining the closed-loop characteristic equation which places the closed-loop poles at locations given by $T(z^{-1})$ below.

$$A(z^{-1})R(z^{-1}) + z^{-1}S(z^{-1})B(z^{-1}) = T(z^{-1}) \qquad (21)$$

where T is a monic fifth order FIR polynomial. Expanding equation 21 and equating like powers of z^{-1} yields the following expressions for the coefficients of S:

$$s_0 = \frac{t_1 - a_1 + 2\cos(wt_s) + 1}{b_0}$$
$$s_1 = \frac{t_2 + (a_1 - 1)(1 + 2\cos(wt_s)) - a_2}{b_0}$$
$$s_2 = \frac{t_3 + (a_2 - a_1)(1 + 2\cos(wt_s)) + 1}{b_0}$$
$$s_3 = \frac{a_1 + t_4 - a_2(1 + 2\cos(wt_s))}{b_0}$$
$$s_4 = \frac{t_5 + a_2}{b_0} \qquad (22)$$

Rearranging equation 16 in terms of the controller gains, K_p, K_i, K_e, K_d, substituting for these gains in equation 20 and using the s_i coefficients of 22 results in the identity of 23.

$$t_1 + t_2 = at_3 + nt_4 + ct_5 + 1 - 2cos(wt_s) \qquad (23)$$

where

$$a = 1 - 2cos(wt_s)$$
$$n = 1 + 2cos(wt_s) - 4cos^2(wt_s)$$
$$c = -1 + 4cos(wt_s) + 4cos^2(wt_s) - 8cos^3(wt_s)$$

Since the MV controller is the stochastic equivalent of the One Step Ahead Controller all the closed-loop poles will be placed at the open-loop zeros. For the process as defined by equation 14 there are no open-loop zeros and consequently $t_1=t_2=\ldots=t_5=0$ and equation 23 reduces to:

$$0.5 = \cos(wt_s)$$

which will not be true in general. This implies that it is not, in general, possible to design a PID controller

with an exact MV interpretation to reject periodic disturbances. An approximate design may be achieved by applying a least squares analysis to equation 19 to obtain the best compromise solution for the controller gains. If such a design is attempted then the closed-loop poles will be placed elsewhere with possible loss of stability. It must be stressed that as equation 23 only applies to the case where periodic disturbances are to be rejected the MV interpretation may be successfully used for the aperiodic case. If, for example, a ramp disturbance is to be rejected equation 19 reduces to four equations in four unknowns which is generally solvable and therefore the design may proceed unhindered.

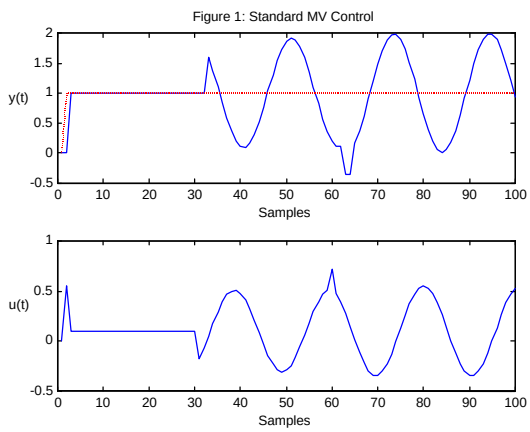
In addition to the previously discussed problem, the PID pole-placement controller suffers from another difficulty that is independent of the chosen disturbance model. If equations 11 and 6 are compared it is evident that with the PID structure additional closed-loop zeros appear in positions defined by the controller numerator, $S(z^{-1})$. These additional zeros are highly undesirable, as their effect on the closed-loop response is unpredictable, but it may safety be presumed that they will detract from the idealised MV response. This problem has previously been discussed by the authors (though in a different context) and two possible solutions are presented by O' Mahony & Downing [5].

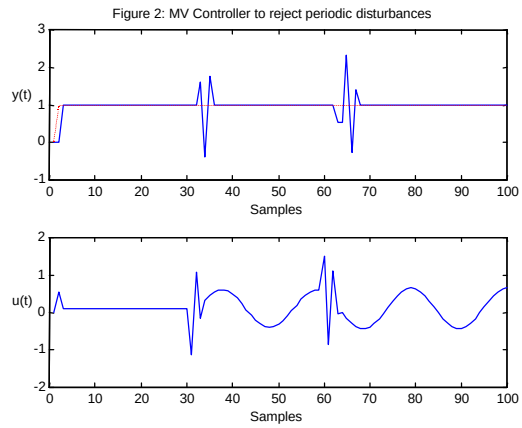
4. Simulation Results

To illustrate the ideas discussed so far the MV and the modified PID controllers were applied to the first order process defined by:

$$G_p(s) = \frac{200}{s+20} \qquad (24)$$

which was sampled at 0.01 sec. The closed-loop was disturbed by sinusoidal disturbances of frequency 30 rads^{-1} applied at both the controller and process output. The sinusoidal disturbance at the controller output, $d_1(t)$, was applied after 30 samples while that on the process output was applied after 60 samples. Both disturbances were of unit peak-to-peak amplitude. The MV controller was first designed by solving the Diophantine equation 7 for F and G given $D(z^{-1}) = (1 - z^{-1})(1 - 2\cos(wt_s)z^{-1} + z^{-2})$ and with the A and B polynomials defined by the discrete equivalent of equation 24. Figures 1 and 2 below illustrate the performance of the modified DPP controller compared with the standard DPP controller in the presence of periodic disturbances.





A modified PID design may be attempted by solving equation 7 for $F(z^{-1})$ and obtaining $S(z^{-1})$ from 18 given A, B as already specified and $R = D$. If equation 19 is then solved, using the least squares approach, with the assumption that $K_d = 0$ (derivative action will only occur for a second order process) the remaining controller gains are found to be $K_p=0.5616$, $K_i=0.0957$ and $K_e=1.2509$. Substituting these values back into the controller equation of 13 and resolving to determine the closed-loop characteristic equation yields closed-loop poles at $z = 0.9935$, $z = -0.5610$, $z=-0.0814\pm0.5928i$. The closed-loop response will obviously not be MV and will be dominated by the pole at $z=0.9$. As was previously mentioned this problem is a direct result of the type of disturbance being rejected and will not occur if the disturbance model is a step or ramp function.

5. Conclusion

This paper has addressed the issue of rejecting general deterministic disturbances be they step, ramp or periodic by nature and presented systematic design approaches for both the minimum variance controller

and a modified PID controller to enable them to reject periodic and aperiodic disturbances. It has been shown that, the chosen tuning methodology for the PID controller fails if the system is subjected to periodic disturbances, and the resulting controller may be unstable. If the controller is stable the desired MV response to set-point will not be achievable, though the disturbance will be rejected. An alternative tuning technique, which overcomes this problem, is presented in [6]. In addition it was illustrated how the traditional PID controller may, under strict assumptions regarding the nature of the process, be tuned as a MV controller and consequently be used to control systems subjected to stochastic disturbances. Finally an example was presented which illustrated the successful rejection of periodic disturbances by the conventional MV controller and the problems that may occur with the modified PID structure.

References:

[1] Åström, K.J., & B. Wittenmark, (1973), 'On self-tuning regulators', *Automatica*, 9, (2), pp. 185-199.

[2] Clarke, D.W., & P.J. Gawthrop, (1979), 'Self-tuning control', *Proceedings IEE*, Vol. 126, 633-639.

[3] Gawthrop, P.J., (1977), 'Some interpretations of the self-tuning controller', *Proceedings IEE*, Vol. 124, 889-894.

[4] O' Dwyer, A, (1996), 'The estimation and compensation of processes with time delays', *Ph. D. Dissertation*, Dublin City University, Dublin 9, Ireland.

[5] O' Mahony, T., & C.J. Downing (1999), 'PID & GPC: A comparative study', *submitted to Irish Systems and Signal Conference*, Galway, Ireland.

[6] O' Mahony, T., & C.J. Downing, (1999), 'Periodic disturbance rejection - Part II: pole-placement control', *submitted to the 3rd IMACS/IEEE International Multiconference on: Circuits, Systems, Communications and Computers*, Athens, Greece.

[7] Tuffs, P.S., & D.W. Clarke, (1985), 'Self-tuning control of offset: a unified approach', *IEE Proceedings D*, 132, (3), pp. 100-110.

[8] Wellstead, P.E., D. Prager, & P. Zanker, (1979), 'Pole assignment self-tuning controller', *Proc. IEE*, 126, (8), pp. 781-787.