Periodic Disturbance Rejection: Part II - Pole-Placement Control†

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Abstract:- Following on from Part I, this paper analyses the capabilities of the pole-placement controller with regard to rejecting deterministic disturbances such as ramp and sinusoidal functions and develops a design strategy to cater for such disturbances. Subsequently the tuning of the extended PID controller of Part I is again addressed, this time from a pole-placement perspective. A technique is developed which is suitable for tuning the extended PID controller in the presence of step, ramp or sinusoidal disturbances. For periodic disturbance rejection restrictions are shown to exist on the possible position of the closed-loop poles, though in most cases satisfactory closed-loop performance should be achievable. The algorithms were tested in simulation and shown to completely reject the sinusoidal disturbance at a single frequency. The various algorithms derived where subsequently shown to be robust to the a-priori knowledge of this frequency. *CSCC'99 Proceedings,* Pages:1901-1907

Keywords:- Pole-Placement Control, PID Control, Deterministic Disturbance Rejection, Robustness analysis.

1. Introduction

In the preceding paper, [7], a procedure for designing a minimum variance controller to reject deterministic disturbances such as ramps and sinusoids was developed. In tandem, the capabilities of the PID controller were also examined and shown to be deficient in this regard. An extended PID controller was developed which was capable of rejecting periodic and aperiodic disturbances. To tune this controller it was proposed to use a minimum variance objective function, which was minimised to yield the controller polynomials. This tuning procedure is applicable for aperiodic disturbances but was shown to yield poor results if the loop was perturbed by periodic disturbances. This paper redresses this issue by considering an alternative tuning technique.

The paper begins by considering the popular pole-placement technique, first presented by Edmunds [4] and later extensively developed by Wellstead and his co-workers [10, 11]. This controller is unable to reject deterministic disturbances and various *ad hoc* techniques such as forcible cascade with an integrator, [8], were used to ensure zero steady state offset. Tuffs [9] overcame this problem by deriving the pole-placement controller on the basis of a CARIMA model rather than the tradition CARMA model. This paper extends this work to enable general deterministic disturbances such as ramps and sinusoids to be rejected by the pole-placement controller. The pole-placement technique is then examined to assess its suitability as a tuning technique for the extended PID controller. It is shown that for step and ramp disturbance function the pole-placement technique can easily be applied and exact pole-placement control is possible, with the poles lying anywhere in the Z-plane. The rejection of sinusoidal disturbances is again problematic as the poles may only be placed in a restricted region of the Z-plane. However by judicious choice of pole locations adequate closed-loop performance is achievable. For the case of periodic disturbances the two algorithms were tested for sensitivity to *a priori* knowledge of the disturbance frequency and shown to be robust.

2. DPP Controller

† This research was funded by the CIT Scholarship Scheme. The authors wish to gratefully acknowledge this support

The Diophantine Pole Placement (DPP) controller is a popular discrete time approach to the control of SISO systems because of the possibility of specifying the desired closed-loop response to set-point. In addition the DPP controller does not suffer from the problem associated with the cancellation of non-minimum phase zeros (except when formulated as a One Step Ahead Controller (OSAC)). It can also easily cater for systems with large time delay as the time delay is incorporated into the plant model. The traditional DPP controller is based on a process model of the type:

$$A(z^{-1})y(t)=z^{-k}B(z^{-1})u(t) \qquad (1)$$

where $A(z^{-1})$ is a monic FIR polynomial of order na, $B(z^{-1})$ is a FIR polynomial of order nb, k is the system time delay, $y(t)$ is the process output and $u(t)$ is the control signal. The controller polynomials are obtained by solving the Diophantine equation

$$A(z^{-1})F(z^{-1}) + z^{-k}B(z^{-1})G(z^{-1}) = T(z^{-1}) \qquad (2)$$

where $F(z^{-1})$ and $T(z^{-1})$ are both monic FIR polynomials of order nf and nt respectively and G is a FIR polynomial of order ng. In equation 2 $G(z^{-1})$ and $F(z^{-1})$ are the controller polynomials and $T(z^{-1})$ is a user-defined polynomial specifying the position of the closed-loop poles. A minimum degree solution for F, and G in eqn. 2 is obtained by choosing the following polynomial degrees:

$$ng = max(na -1, nt - nb - k);$$
$$nf = nb + k -1;$$

and equating the coefficients of like powers of z^{-1} to obtain a set of simultaneous equations which can then be solved. For a solution to eqn. 2 to exist it is also required that either A and B be co-prime or that the polynomial T also contains the common factor. Given the controller polynomials F and G the following linear control law may be implemented:

$$H(z^{-1})r(t) = G(z^{-1})y(t) + F(z^{-1})u(t) \qquad (3)$$

where $r(t)$ is the user specified input, and $H(z^{-1})$ is generally a scalar, H, chosen to ensure zero steady state offset i.e.

$$H = \frac{\lim T(z^{-1})|_{z\to 1}}{\lim B(z^{-1})|_{z\to 1}} = \frac{T(1)}{B(1)}$$

To extend the approach to deal with general deterministic disturbances the Diophantine equation is modified as in the MV approach to yield:

$$A(z^{-1})D(z^{-1})F(z^{-1}) + z^{-k}B(z^{-1})G(z^{-1}) = T(z^{-1}) \qquad (4)$$

which results in the following linear controller

$$H(z^{-1})r(t) = G(z^{-1})y(t) + F(z^{-1})D(z^{-1})u(t) \qquad (5)$$

The polynomial $D(z^{-1})$ models the disturbance and typically consists of either a step, ramp or sinusoidal function. Again a prerequisite is that $A'=AD$ and B be co-prime, or if they are not then T must also include the common factor. A minimum degree solution to equation 4 is obtained by choosing the following polynomial degrees:

$$ng = max(na + nd -1, nt - nb - k)$$
$$nf = nb + k -1$$

where nd is the order of the $D(z^{-1})$ polynomial. Section 4 will illustrate the DPP rejection of sinusoidal disturbances.

In equation 4 the objective of the user chosen $T(z^{-1})$ polynomial is to specify the location of the closed-loop poles. Clearly if this polynomial is chosen such that all the poles lie at the origin of the Z-plane ($T=1$) then the pole-placement controller reduces to a dead-beat controller, in which the output will reach the set-point after n sample periods where n is the order of the process. The closed-loop transfer function relating output to input for the dead-beat controller is:

$$y(t) = \frac{z^{-k}BH}{T(z^{-1})}r(t) = z^{-k}BHr(t)$$

Note that the dead-beat controller does <u>not</u> cancel the process zeros and consequently non-minimum phase processes are not an issue.

The OSAC may also be obtained from equation 4. This controller is characterised by a response that follows the set-point with a single sample period delay. By considering the closed-loop transfer function it is evident that if the closed-loop characteristic polynomial, T, is equated with the process numerator then, in the absence of time delay, OSAC is obtained. Equation 5 becomes:

$$A(z^{-1})D(z^{-1})F(z^{-1}) + B(z^{-1})G(z^{-1}) = B(z^{-1})$$

where

$$F(z^{-1}) = B(z^{-1})$$
$$G(z^{-1}) = [1 -A(z^{-1})D(z^{-1})]$$

This yields the following control law

$$H(z^{-1})r(t) = [1 - A(z^{-1})D(z^{-1})]y(t) + B(z^{-1})D(z^{-1})u(t)$$

Since the controller poles contain the process zeros, this controller cannot be used with non-minimum phase processes. In addition, if a time delay of k samples exists, the process will be unable to respond until at least $k+1$ samples later and this controller becomes a $k+1$ step ahead controller.

3. PID Control

In the companion paper by the same authors' [7], it was shown that the standard PID controller does not have the capability to completely reject ramp or periodic disturbances. It was consequently shown that to incorporate deterministic disturbance rejection it was necessary to modify the PID controller as in equation 6.

$$u(t) = \left[K_p + \frac{K_i}{(1-z^{-1})} + \frac{K_e}{D'(z^{-1})} + K_d(1-z^{-1})\right]\mathrm{e(t)} \quad (6)$$

where K_e is an additional gain term associated with the disturbance. Since this controller already incorporates an integrator $D'(z^{-1})$ will typically model either a ramp or sinusoid function:

- Ramp $D'(z^{-1}) = (1 - z^{-1})^2$
- Sinusoid $D'(z^{-1}) = (1 - 2\cos(wt_s)z^{-1} + z^{-2})$ (7)

As noted by Isermann [5] tuning techniques for high performance PID controllers may be divided into methods which minimise some performance criterion and methods which place the closed-loop poles at some desirable locations in the Z plane. In the previous paper the former design approach was discussed and it was noted that problems arose when sinusoidal disturbances were applied at the process input or output. This section will demonstrate a pole-placement tuning technique for the extended PID controller, though it can be equally applicable to the standard three-term controller.

To illustrate the pole-placement design philosophy it is assumed that a controller structure, designed to reject offset and sinusoidal disturbances, as defined by equation 8 is desirable.

$$u(t) = \left[K_p + \frac{K_i}{(1-z^{-1})} + \frac{K_e}{\left(1-2\cos(wt_s)z^{-1}+z^{-2}\right)} + K_d(1-z^{-1})\right]\mathrm{e(t)} \quad (8)$$

Since this modified PID controller still retains a strict structure there are limitations imposed on the maximum order of the process model that can be used. These restrictions are identical to those for the standard PID controller and are necessary to ensure the exact placement of the closed-loop poles [12] i.e.

$$A(z^{-1}) = 1 + a_1z^{-1} + a_2z^{-2}; \qquad B(z^{-1}) = b_0; \quad k=1; \quad (9)$$

Equation 8 can equivalently be represented in difference equation form as,

$$u(t) = \frac{s_0 + s_1z^{-1} + s_2z^{-2} + s_3z^{-3} + s_4z^{-4}}{1 + r_1z^{-1} + r_2z^{-2} + r_3z^{-3}}e(t) \quad (10)$$

where

$$\begin{aligned}
s_0 &= K_p + K_i + K_e + K_d \\
s_1 &= -K_p(1+2b) - 2bK_i - K_e - K_d(2+2b) \\
s_2 &= K_p(1+2b) + K_i + 2K_p(1+2b) \\
s_3 &= -K_p - K_d(2+2b) \\
s_4 &= K_d \\
r_1 &= -(1+2\cos(wt_s)) \\
r_2 &= 1+2\cos(wt_s) \\
r_3 &= -1
\end{aligned} \quad (11)$$

and $b = \cos(wt_s)$. Combining equations 9 and 10 yields the following expression for the closed-loop characteristic equation which needs to be solved to yield the controller numerator $S(z^{-1})$

$$A(z^{-1})R(z^{-1}) + z^{-1}B(z^{-1})S(z^{-1}) = T(z^{-1}) \quad (12)$$

This is a limited version of equation 4 where

$$R(z^{-1}) = D(z^{-1}); \quad S(z^{-1}) = G(z^{-1}); \quad F(z^{-1}) = 1 \quad (13)$$

and can be solved by comparing like terms of z^{-1}. Because of the restrictions imposed by equation 9 the user-chosen $T(z^{-1})$ polynomial is limited to at most fifth order:

$$T = 1 + t_1z^{-1} + t_2z^{-2} + t_3z^{-3} + t_4z^{-4} + t_5z^{-5}$$

From the preceding analysis it is obvious that obtaining the R and S polynomials is a relatively straightforward procedure. However in practice it will be desirable to implement equation 8 to provide the operator with the freedom to modify the closed-loop performance by varying the controller gains. Consequently it will be necessary to compute the controller gains (K_p, K_i, K_e, K_d) from the controller numerator $S(z^{-1})$. Examining the set of equations 11 it is obvious that this system of linear equations is overdetermined and for an exact solution to exist the following identity must be satisfied [7]:

$$t_1 + t_2 = at_3 + nt_4 + ct_5 + 1 - 2\cos(wt_s) \quad (14)$$

where

$$\begin{aligned}
a &= 1 - 2\cos(wt_s) \\
n &= 1 + 2\cos(wt_s) - 4\cos^2(wt_s) \\
c &= -1 + 4\cos(wt_s) + 4\cos^2(wt_s) - 8\cos^3(wt_s)
\end{aligned}$$

This identity has two implications.

1. If the controller is designed by initially deciding

on a suitable T polynomial, and subsequently the controller gains of equation 8 are to be determined from 11, then in general exact values of K_p, K_i, K_e, and K_d will not be available. It is possible however, to find the best compromise solution, the one that comes closest to satisfying all equations simultaneously. If closeness is defined in the least squares sense, i.e. that the sum of the differences between the left- and right-hand sides of equation 11 be minimised, then the overdetermined linear problem reduces to a (usually) solvable linear problem and the controller gains can then be obtained.

2. Once the controller gains have been defined, either by using (1) above or any alternative tuning technique e.g. trial-and-error, then equation 8 may be used to implement the controller. By implementing 8 it naturally follows that the identity 14 has to be satisfied and consequently the possible positions of the closed-loop poles are restricted in accordance with 14.

The net effect of this is that the user specified polynomial $T(z^{-1})$ will only approximately define the closed-loop characteristic equation and in many cases it may well be possible that the desired closed-loop response is unattainable. These points will be clarified, via an example, in the following section. It should also be stressed that this problem only exists when the controller is structured to reject periodic disturbances. If, for example, ramps are to be rejected then the equivalent version of 11 will reduce to a system with four equations in four unknowns which is usually solvable. This implies that for aperiodic disturbances the characteristic equation may be exactly specified by a $T(z^{-1})$ polynomial whose roots may lie anywhere in the Z-plane.

Another problem which afflicts the PID pole-placement controller is that additional closed-loop zeros appear in positions defined by the controller numerator, $S(z^{-1})$, which do not appear in the DPP structure. These additional zeros are highly undesirable, as their effect on the closed-loop response is unpredictable. The authors' have previously discussed this problem and two possible solutions are presented in [6].

4. Simulation Results

To illustrate the ideas discussed so far the DPP and the modified PID controllers were applied to the first order process defined by:

$$G_p(s) = \frac{200}{s+20} \qquad (15)$$

which was sampled at 0.01 sec. The closed-loop was disturbed by sinusoidal disturbances of frequency 30 rads^{-1} applied at both the controller and process output. The sinusoidal disturbance at the controller output, $d_1(t)$, was applied after 30 samples while that on the process output was applied after 60 samples. Both disturbances were of unit peak-to-peak amplitude. The DPP controller was first designed by solving the Diophantine equation 4 for F and G given that the closed-loop poles were specified by:

$$T(z^{-1}) = 1 - 1.584z^{-1} + 0.657z^{-2}$$

and

$$D(z^{-1}) = (1-z^{-1})(1 - 2\cos(wt_s)z^{-1} + z^{-2})$$

with the A and B polynomials defined by the discrete equivalent of equation 15. Figures 1 and 2 below illustrate the performance of this DPP controller compared with the standard DPP controller in the presence of periodic disturbances.

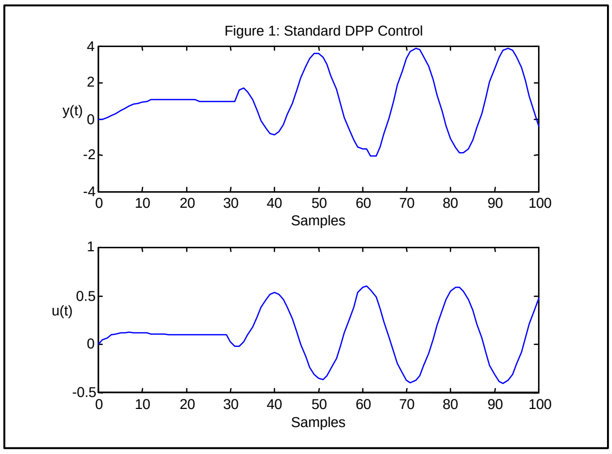


Figure 1: Standard DPP Control

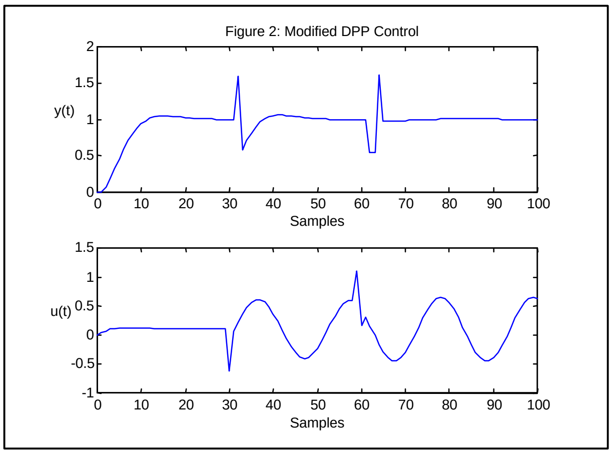
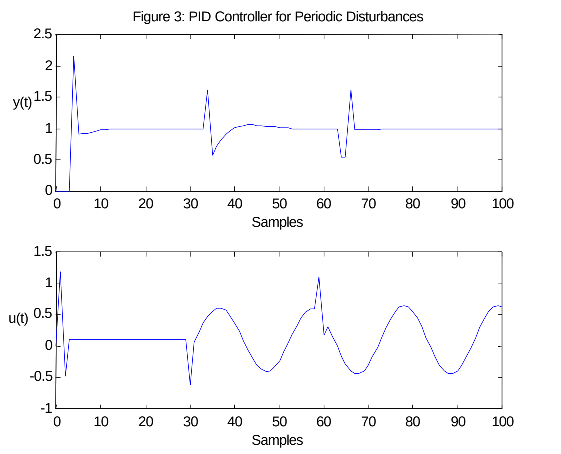


Figure 2: Modified DPP Control

The PID controller was similarly designed by solving equation 12 for $S(z^{-1})$ given A, B, T as already specified and with $R(z^{-1}) = D(z^{-1})$. This yields a solution for S which is identical to that found for G in the DPP case. To obtain the controller gains it has already been noted that the identity 14 must be satisfied. For this specific case 14 reduces to

$$t_1 + t_2 = 1 - 2\cos(wt_s) \qquad (16)$$

which in this case is approximately, though not exactly, true. Consequently it is not possible to find an analytical solution to the set of equations as defined by 11. If the linear least squares solution to 11 is sought the controller gains are found to be $K_p=0.45$, $K_I=0.57$, $K_e=0.17$, $K_d=0.001$. The response obtained when this controller was implemented is illustrated in figure 3. Comparing figures 2 and 3 it is apparent that the disturbance responses are very similar but that the set-point responses differ radically. In the extended PID controller the set-point response it characterised by a large overshoot, which is a direct result of the controller zeros appearing in the closed-loop equation. If these additional zeros are cancelled the extended PID response will be identical to that illustrated in figure 2.



5. Additional Remarks

It is of considerable interest to determine the extent to which equation 14 restricts the possible locations of the closed-loop poles. If, as in the preceding example, a second order $T(z^{-1})$ polynomial is assumed:

$$T(z^{-1}) = 1 + t_1 z^{-1} + t_2 z^{-2}$$

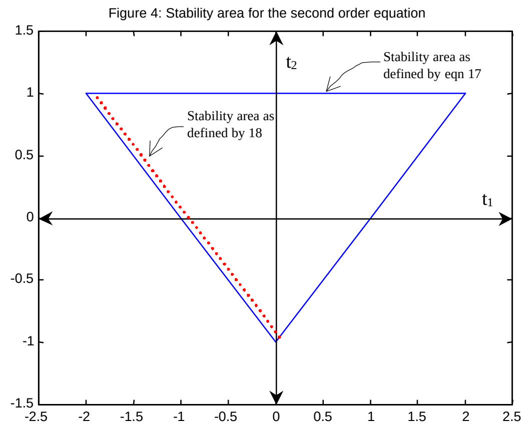
then by applying Jury's stability test [1] it can be shown that the region of stability for a second order equation is defined by:

$$\begin{aligned} t_2 &< 1 \\ t_2 &> -1 + t_1 \\ t_2 &> -1 - t_1 \end{aligned} \qquad (17)$$

Using equation 16 the above can be re-written as:

$$\begin{aligned} t_2 &< 1 \\ t_2 &> -\cos(wt_s) \\ t_1 + t_2 &= 1 - 2\cos(wt_s) \end{aligned} \qquad (18)$$

Graphically the stability area as defined by equations 17 and 18 may be illustrated as in figure 4, for $w = 30rad/s$ and $t_s = 0.01sec$.



Further insight may be obtained by considering the permissible region for the closed-loop poles in the Z-plane. In general there exists two possibilities for the closed-loop poles; either both poles will be real, or both will be complex and form a complex conjugate pair. Considering the case of two real poles i.e.

$$\begin{aligned} T(z^{-1}) &= (z-a)(z-b) \\ &= z^2 - z(a+b) + ab \end{aligned} \qquad (19)$$

and comparing coefficients yields

$$\begin{aligned} t_1 &= -(a+b) \\ t_2 &= ab \end{aligned}$$

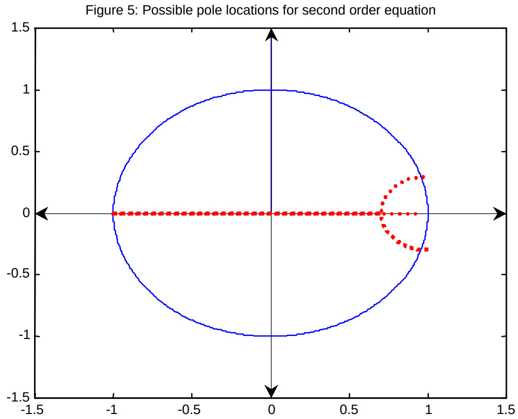
By substituting for t_1 and t_2 in equation 18 it can be shown that for a stable solution to exist equation set 20 must be satisfied.

$$\begin{aligned} ab &< 1 \\ ab &> -\cos(wt_s) \\ -1 < a &< \cos(wt_s) \\ -1 < b &< \cos(wt_s) \end{aligned} \qquad (20)$$

Applying a similar strategy to the complex poles case results in the following additional criteria which must also be satisfied:

$$\begin{aligned} a^2 + b^2 &< 1 \\ (a-1)^2 + b^2 &= 2 - 2\cos(wt_s) \end{aligned} \qquad (21)$$

where the complex conjugate poles are defined by $z = a \pm jb$. For this particular case the possible positions for the closed-loop poles are as illustrated in figure 5 (heavy, dotted line). While equation 16 imposes significant restrictions on the possible closed-loop pole locations, it is also evident that sufficient reasonable choices remain to enable satisfactory closed-loop performance.



Thus far it has been illustrated how a discrete time controller may be derived which will successfully reject a periodic disturbance of known frequency. In reality, though, it is unlikely that the frequency of the periodic disturbance will be exactly known or it may have to be estimated in which case there is likely to be some uncertainty regarding the exact frequency. Consequently the development thus far will be of little practical use unless it can be shown that the modified controllers are robust to uncertainty regarding the frequency of the periodic disturbance. To evaluate this the process as defined by equation 15 was again used. If a sampling period of $t_s=0.01sec$ is used then, the highest practical frequency permissible according to Nyquist will be

$$w_d = \frac{2p}{5t_s} = 125 rads^{-1}$$

An appropriate anti-aliasing filter will ensure that this is so in practice. Again a unit peak-to-peak periodic disturbance was applied to both the process and controller outputs. The controllers were designed on the assumption that the nominal frequency of the disturbance was 30rads^{-1}, while in the simulation the actual frequency of the disturbance was varied according to the first column of table 1 (next page). The effect of the disturbance on the output was tabulated in terms of the peak-to-peak ripple recorded on the output. In table 1, four different controllers are examined. These are the DPP, MV, PID and GPC controller of Clarke *et al* [2, 3]. Two different objectives were examined in most cases. These are labelled DB (Deat-beat) and PP (pole-placement with T=1-1.584z^{-1}+ 0.657z^{-2}). In addition the GPC controller has a default parameter set which was also investigated. Clearly the dead-beat controllers performed best with over 95% attenuation in a ±15% band. These results confirm that the algorithms are robust to the choice of frequency. From table 1 it is also obvious that most of the controllers are able to successfully attenuate low frequency disturbances to a greater extent than high frequency disturbances, which is not surprising considering the nature of the process.

6. Conclusion

This paper has presented some guidelines for designing pole-placement controllers that can reject both periodic and aperiodic disturbances. An alternative tuning method for the extended PID controller based on pole-placement was also presented. If the strict assumptions regarding the nature of the process are adhered to, an exact pole-placement interpretation for the PID controller is possible even if the process is disturbed by aperiodic disturbances. The PID pole-placement design for a process perturbed by periodic disturbances is not as simple and care must be taken when specifying the positions of the closed-loop poles. However it has been illustrated that the limitations introduced as a result of equation 14 are not overly restrictive and in most cases satisfactory closed-loop performance should be achievable. Finally it has also been demonstrated, through a simulation example that exact rejection of periodic disturbances is possible if the frequency of the disturbance is known *a priori*. Subsequently it has illustrated that the various algorithms are robust to exact knowledge of this frequency.

Table 1: Robustness of Controllers to variations in Frequency							
Frequency (rad/s)	***DPP Controller***		***GPC***			***PID***	***MV***
	PP	DB	Default	PP	DB	PP	
1	0.0244	0.0018	0.0999	0.0244	0.0018	0.0244	0.0018
10	0.2156	0.158	0.9106	0.2156	0.158	0.2156	0.158
20	0.1240	0.0196	1.3181	0.1240	0.0196	0.1240	0.0196
25	0.1529	0.0134	1.0014	0.1529	0.0134	0.1529	0.0134
29	0.0338	0.0033	0.2353	0.0338	0.0033	0.0338	0.0033
30	1.03e-13	1.5e-13	8.12e-6	1.03e-13	1.59e-13	1.03e-13	1.54e-13
31	0.0348	0.0037	0.2316	0.0348	0.0037	0.0348	0.0037
35	0.1807	0.0218	0.9793	0.1807	0.0218	0.1807	0.0218
40	0.3657	0.0532	1.4863	0.3657	0.0532	0.3657	0.0532
50	0.7095	0.1487	1.8788	0.7095	0.1487	0.7095	0.1487
60	1.0081	0.2928	2.0163	1.0081	0.2928	1.0081	0.2928
100	1.8655	1.3930	2.0383	1.8655	1.3930	1.8655	1.3930

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