The Generalized Discrete-Time Riccati Equation : Towards a new explicit matrix block formulation

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Abstract: - Generalized discrete time Riccati equation is analyzed from a matrix block formulation point of view. Some recently published results are firstly discussed. Then a new multi-blocks equation is introduced, from which follows a quite general updating formula. A new class of iterative algorithms is now available through a unique linear system resolution by step. Convergence conditions are set up in terms of stabilisabilty and detectability.
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1 Introduction

This paper deals with different matrix formulations for the generalized discrete-time Riccati equations, (G-DARE). These equations arise in many areas, as for example in control theory, [1] [2]. Within this framework, the linear quadratic optimal regulator problem for the time-invariant discrete-time descriptor system

$$\begin{cases} \mathbf{Ex}(k+1) = \mathbf{Ax}(k) + \mathbf{Bu}(k) \\ \mathbf{y}(k) = \mathbf{Cx}(k) + \mathbf{Du}(k) \end{cases} \qquad (1)$$

consists in computing a control **u = K.x** which minimizes the cost functional

$$\begin{aligned} J(\mathbf{Ex}(0), u, N) &= \frac{1}{2}\mathbf{x}(N)\mathbf{E}^T\mathbf{P}_N\mathbf{Ex}(N) \\ &+ \frac{1}{2}\sum_{k=0}^{N-1}\begin{bmatrix}\mathbf{x}(k)\\ \mathbf{u}(k)\end{bmatrix}^T\begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^T & \mathbf{R}\end{bmatrix}\begin{bmatrix}\mathbf{x}(k)\\ \mathbf{u}(k)\end{bmatrix} \end{aligned} \qquad (2)$$

where $\mathbf{x} \in \Re^n$ and $\mathbf{u} \in \Re^m$, and **A, B, Q, S, R** are matrices of appropriate dimensions.

As usually, it is assumed that the global cost matrix is symmetric and positive semi-definite :

$$\begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^T & \mathbf{R}\end{bmatrix} = \begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^T & \mathbf{R}\end{bmatrix}^T \geq \mathbf{0}$$

and that the pencil $(z\mathbf{E} - \mathbf{A})$ is regular (i.e. $|(z\mathbf{E} - \mathbf{A})| \neq 0$). Here, only the infinite time horizon problem is considered. Basically, **E** is here restricted to be non singular. Now, under technical conditions, specified latter, it is well known that the minimizing control **u** which is stabilizing (i.e. the generalized eigenvalues of $[\lambda\mathbf{E} - (\mathbf{A} + \mathbf{BK})]$ have modulus less than one) is given by:

$$\mathbf{u}(k) = -(\mathbf{R} + \mathbf{B}^T\mathbf{XB})^{-1}(\mathbf{B}^T\mathbf{XA} + \mathbf{S}^T)\mathbf{x}(k) \qquad (3)$$

where **X** is the unique nonnegative definite solution of the so called "standard" G-DARE :

$$\begin{aligned} \mathbf{E}^T\mathbf{XE} &= \mathbf{A}^T\mathbf{XA} + \mathbf{Q} \\ &- (\mathbf{A}^T\mathbf{XB} + \mathbf{S})(\mathbf{R} + \mathbf{B}^T\mathbf{XB})^{-1}(\mathbf{B}^T\mathbf{XA} + \mathbf{S}^T) \end{aligned} \qquad (4)$$

The paper is organized as follow. Section 2 and 3 review the Bernard & al [3] and the Nikoukhah [4] [5] results. Those papers provide some interesting matrix blocks equations initially developed for the singular case, i.e. when **E** is singular. Section 4 deals with our new formulation, a 5-blocks matrix equation, and its connection with Bernhard's. and Nikoukhah's previous work. The standard technical conditions of stabilisability and detectability are reviewed. A new iterative algorithm is deduced. Lastly, section 5 gives some concluding remarks and open questions.

2 A 2-blocks matrix formulation [3]

Bernhard & al. consider the infinite-time quadratic optimal regulator problem associated with the cost functional :

$$\begin{aligned} J &= \frac{1}{2}\sum_{k=0}^{\infty}\begin{bmatrix}\mathbf{x}(k)\\ \mathbf{u}(k)\end{bmatrix}^T\begin{bmatrix}\mathbf{Q} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}\end{bmatrix}\begin{bmatrix}\mathbf{x}(k)\\ \mathbf{u}(k)\end{bmatrix} \\ &\equiv \frac{1}{2}\sum_{k=0}^{\infty}\mathbf{y}^T(k).\mathbf{y}(k) \end{aligned} \qquad (5)$$

In this paper, **R** is restricted to be regular, although (1-2) remain well define, even if **R** is singular [6]. However, **E** can be singular. Note also that it has been taken $\mathbf{C}^{\mathrm{T}}\mathbf{D}=\mathbf{0}$, hence $\mathbf{S}=\mathbf{0}$. Consequently, when **E** is regular, (4) is reduced to :

$$\begin{aligned}\mathbf{E}^{\mathbf{T}}\mathbf{X}\mathbf{E} &= \mathbf{A}^{\mathbf{T}}\mathbf{X}\mathbf{A}+\mathbf{Q}\\ &-\mathbf{A}^{\mathbf{T}}\mathbf{X}\mathbf{B}\left(\mathbf{R}+\mathbf{B}^{\mathbf{T}}\mathbf{X}\mathbf{B}\right)^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{X}\mathbf{A}\end{aligned} \tag{7}$$

In this particular context, we have :

2.1 Definition 1 [3]

The descriptor system (1) is said to be stabilisable if for all complex $(\lambda,\mu)\neq(0,0)$, the matrix $[\lambda\mathbf{E}-\mu\mathbf{A},\ \ \mathbf{B}]$ is full row rank for $|\lambda|\geq|\mu|$, i.e. $[\mathbf{E},\ \ \mathbf{B}]$ is full row rank and $[\lambda\mathbf{E}-\mathbf{A},\ \ \mathbf{B}]$ is full row rank for all $|\lambda|\geq 1$.

2.2 Definition 2 [3]

The descriptor control problem(1) and (5) is said to be detectable if for all complex $(\lambda,\mu)\neq(0,0)$, the matrix $\begin{bmatrix}\lambda\mathbf{E}-\ \ \mu\mathbf{A}\\ \mathbf{C}\end{bmatrix}$ is full column rank for $|\lambda|\geq|\mu|$, i.e. $\begin{bmatrix}\mathbf{E}\\ \mathbf{C}\end{bmatrix}$ is full column rank and $\begin{bmatrix}\lambda\mathbf{E}-\mathbf{A}\\ \mathbf{C}\end{bmatrix}$ is full column rank for all $|\lambda|\geq 1$.

Remarks : The same is true with **C** replaced by **Q.** Remember that **E** can be singular**.**

2.3 Theorem 1 [3]

Let the singular descriptor control problem (1) and (5) be stabilisable and detectable. Then the quadratic optimal regulator problem has a unique solution (3) (with $\mathbf{S}=\mathbf{0}$ and **R** > **0**) where **Y** is the unique nonnegative solution of the matrix 2-blocks equation

$$\mathbf{Y}=\begin{bmatrix}\mathbf{0}\\ \mathbf{A}\end{bmatrix}^{\mathbf{T}}\mathbf{M}^{-1}\begin{bmatrix}\mathbf{Y} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\end{bmatrix}\mathbf{M}^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{A}\end{bmatrix}+\mathbf{Q} \tag{7a}$$

and $\mathbf{M}=\begin{bmatrix}\mathbf{Y} & \mathbf{E}^{\mathrm{T}}\\ \mathbf{E} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\end{bmatrix}$ (7b)

with $\mathbf{E}^{\mathbf{T}}\mathbf{X}\mathbf{E}=\mathbf{Y}$ (7c)

This first block matrix equation can be further reduced to [5] :

$$\mathbf{Y}=-[\mathbf{0}\quad \mathbf{A}]^{\mathrm{T}}\mathbf{M}^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{A}\end{bmatrix}+\mathbf{Q} \tag{8}$$

Lastly, after a few manipulations on equation (8), and if **Y** is regular, it is easy to see that :

$$\mathbf{Y}=\mathbf{A}^{\mathbf{T}}\left(\mathbf{E}\mathbf{Y}^{-1}\mathbf{E}^{\mathbf{T}}+\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\right)^{-1}\mathbf{A}+\mathbf{Q} \tag{9}$$

If $\mathbf{E}=\mathbf{1}$, then (9) is here equivalent to the standard Riccati equation (4) with $\mathbf{S}=\mathbf{0}$ and **R** regular.

However, (3) and (4) define the unique solution of the control problem, under the hypothesis of stabilisability and detectability, so $\mathbf{Y}=\mathbf{E}^{\mathbf{T}}\mathbf{X}\mathbf{E}$ given by (8) is such that **X** is also the solution of (4), even if matrices **X** and **Y** are singular while matrix **E** remains non singular.

From (8), follows also a first iterative matrix block algorithm :

$$\mathbf{Y}_{\mathbf{k+1}}=-[\mathbf{0}\quad \mathbf{A}]^{\mathbf{T}}\mathbf{M}_{\mathbf{k}}^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{A}\end{bmatrix}+\mathbf{Q},\quad \mathbf{Y}_{\mathbf{0}}=\mathbf{Q} \tag{10}$$

whose limit is $\mathbf{Y}=\mathbf{E}^{\mathrm{T}}\mathbf{X}\mathbf{E}$ and **X** is the unique positive semi-definite solution of (4).

Although these results do not involve the generalized cost function (2) (remember that **R** is regular and $\mathbf{S}=\mathbf{0}$), it remains interesting because (8) it can be considered as an alternative to the implicit standard Riccati recursion:

$$\begin{aligned}\mathbf{E}^{\mathrm{T}}\mathbf{X}_{\mathbf{k+1}}\mathbf{E} &= \mathbf{A}^{\mathbf{T}}\mathbf{X}_{\mathbf{k}}\mathbf{A}+\mathbf{Q}\\ &-\left(\mathbf{A}^{\mathbf{T}}\mathbf{X}_{\mathbf{k}}\mathbf{B}+\mathbf{S}\right)\left(\mathbf{R}+\mathbf{B}^{\mathbf{T}}\mathbf{X}_{\mathbf{k}}\mathbf{B}\right)^{-1}\left(\mathbf{B}^{\mathbf{T}}\mathbf{X}_{\mathbf{k}}\mathbf{A}+\mathbf{S}^{\mathbf{T}}\right)\end{aligned} \tag{11}$$

However, (10) no more than (11) give explicitly **X**.

3 A 3-blocks formulation [5]

This Nikoukhah's work [5] deals basically with a general formulation of a discrete-time filtering problem for descriptor systems. However, in a previous report [4], Nikoukhah solves the dual control problem

$$\mathbf{E}\mathbf{x}(k+1)=\mathbf{A}\mathbf{x}(k)+\mathbf{B}\mathbf{u}(k+1),\ \mathbf{x}_0 \text{ given} \tag{12a}$$

$$J=\frac{1}{2}\sum_{k=0}^{\infty}\begin{bmatrix}\mathbf{x}(k)\\ \mathbf{u}(k)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^{\mathbf{T}} & \mathbf{R}\end{bmatrix}\begin{bmatrix}\mathbf{x}(k)\\ \mathbf{u}(k)\end{bmatrix} \tag{12b}$$

The stabilisability remains define as previously. Now, because (12) takes into account the cross-weighted matrix S, the definition of detectability is extended as follows :.

3.1 Definition 3 [4]

The descriptor control problem (12) is said detectable if $\begin{bmatrix} \lambda\mathbf{E}-\mu\mathbf{A} & \mathbf{B} \\ \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{bmatrix}$ has full column rank for all $(\lambda,\mu)\neq(0,0)$ such that $|\lambda|\geq|\mu|$, i.e. $\begin{bmatrix} \mathbf{E} & \mathbf{B} \\ \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{bmatrix}$ is full column rank and $\begin{bmatrix} \lambda\mathbf{E}-\mathbf{A} & \mathbf{B} \\ \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{bmatrix}$ is full column rank for all $|\lambda|\geq 1$.

3.2 Theorem 2 [4]

Suppose that the descriptor control problem (11) is stabilisable and detectable according to definitions 1 and 3. Then, the solution of the infinite time horizon problem is given by

$$\begin{pmatrix} \mathbf{x}(k) \\ \mathbf{u}(k) \end{pmatrix} = \begin{bmatrix} \mathbf{I}\,\mathbf{0}\,\mathbf{0} \\ \mathbf{0}\,\mathbf{I}\,\mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Q}+\mathbf{A}^T\mathbf{X}\mathbf{A} & \mathbf{S} & \mathbf{E}^T \\ \mathbf{S}^T & \mathbf{R} & -\mathbf{B}^T \\ \mathbf{E} & -\mathbf{B} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{x}(k-1)$$

where $\mathbf{X}$ is the unique positive semi-definite solution of the algebraic descriptor Riccati equation

$$\mathbf{X} = -[\mathbf{0}\ \ \mathbf{0}\ \ \mathbf{I}] \begin{bmatrix} \mathbf{Q}+\mathbf{A}^T\mathbf{X}\mathbf{A} & \mathbf{S} & \mathbf{E}^T \\ \mathbf{S}^T & \mathbf{R} & -\mathbf{B}^T \\ \mathbf{E} & -\mathbf{B} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} \quad (16)$$

This is a very appealing result, unfortunately, $\mathbf{X}$ does not solve the standard D-GARE (when $\mathbf{E}$ is regular) as defined classically by (4). In fact, Nikoukhah considers a slightly different control problem (with $\mathbf{Bu}(k+1)$), and his results cannot be directly compared with anything else.

However, following similar ideas, it has been discovered, some new multi blocks matrix equations, which generalize Bernhard's works [3], and give the solution of equation (4).

4 A new 5-blocks formulation

The objective, here, is to find a matrix equation similar (8) which releases the constraints **S = 0**, and **R** regular. The last one is easy to take into account. Using standard block matrix inversion lemmas [5] [7], it can be verified that :

$$\mathbf{Y} = -[\mathbf{0}\ \ \mathbf{0}\ \ \mathbf{A}^T] \begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{B}^T \\ \mathbf{0} & \mathbf{Y} & \mathbf{E}^T \\ \mathbf{B} & \mathbf{E} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{A} \end{bmatrix} + \mathbf{Q} \quad (17)$$

is equivalent to (8). Then (17) is equivalent to (4), when $\mathbf{E}$ is non singular in the same conditions as previously ($\mathbf{Y}=\mathbf{E}^T\mathbf{X}\mathbf{E}$). Under the hypothesis of stabilisability and detectability, the following algorithm converges to the unique symmetric positive semi definite solution $\mathbf{Y}=\mathbf{E}^T\mathbf{X}\mathbf{E}$ where $\mathbf{X}$ is the unique symmetric positive semi-definite solution of equation (4):

$$\mathbf{Y}_{k+1} = -[\mathbf{0}\ \ \mathbf{0}\ \ \mathbf{A}^T] \begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{B}^T \\ \mathbf{0} & \mathbf{Y}_k & \mathbf{E}^T \\ \mathbf{B} & \mathbf{E} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{A} \end{bmatrix} + \mathbf{Q} \quad (18)$$

$$\mathbf{Y}_0 = \mathbf{Q}$$

It seems that there is no “elementary” extension including the cross term **S**. In order to achieve this objective, a completely new 5-blocks structure is introduced. The proof is quite tedious and is not reported here. The main result is as follows.

4.1 Theorem 3

If the control problem

$$\begin{cases} \mathbf{Ex}(k+1) = \mathbf{Ax}(k) + \mathbf{Bu}(k) \\ \mathbf{y}(k) = \mathbf{Cx}(k) + \mathbf{Du}(k) \end{cases} \quad (19a)$$

$$J = \frac{1}{2}\sum_{k=0}^{\infty} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{u}(k) \end{bmatrix}^T \begin{bmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{u}(k) \end{bmatrix} \quad (19b)$$

is stabilizable and detectable, according to definitions 1 and 3, then

$$\mathbf{X} = -[\mathbf{0}\ \ \mathbf{0}\ \ \mathbf{0}\ \ \mathbf{0}\ \ \mathbf{1}] \begin{bmatrix} \mathbf{X} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{S}^T & \mathbf{B}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{S} & \mathbf{Q} & \mathbf{A}^T & \mathbf{E}^T \\ \mathbf{1} & \mathbf{B} & \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{E} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \quad (20)$$

is equivalent to (4), when E is regular.

Furthermore, the following iterative algorithm converges to the unique symmetric positive semi definite solution of (4) :

$$\mathbf{X}_{k+1} = -[\mathbf{0\ 0\ 0\ 0\ 1}]\begin{bmatrix} \mathbf{X}_k & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{S}^T & \mathbf{B}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{S} & \mathbf{Q} & \mathbf{A}^T & \mathbf{E}^T \\ \mathbf{1} & \mathbf{B} & \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{E} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \quad (21)$$

$$\mathbf{X}_0 = \mathbf{Q}$$

As it can be seen, the recursion scheme (20) is the only available algorithm that gives explicitly the matrix **X** solution of (4).

5 Conclusion

A new matrix-block formulation has been introduced as an equivalent equation to the standard D-GARE relation. It leads to an explicit iterative algorithm compared with classical implicit recurrence scheme, when E is not reduced to the unity matrix.

Furthermore, the cross term **S** is naturally embedded in this formulation, while R can remain singular, without any difficulties.

From a numerical point of view, it will be interesting to explore the conditioning of the required block-matrix, versus the conditioning of the Riccati equation itself.

Because of the special structure, which appears in (21), very efficient computations can be achieved. It is only necessary to compute the last (5,5) block of the inverse, which in turn can be obtained in solving the corresponding linear system whose right hand side is the last n columns of the unity matrix of appropriate size.

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