# How to choose membership functions for fuzzy models in approximation problems

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*Abstract:* This paper shows the conditions needed to be able to determine the behavior of the output of a fuzzy model from the input variables membership functions. This is made possible by breaking down the model into sections and in the relation between the fuzzy sets and the fuzzy rules for each section. Through this procedure, we obtain a well-defined relation between the inputs and the output of the model. We present the results for the approximation of a maximum (minimum) of a fuzzy model with two inputs, and we give the results for the general case of multi-inputs. Besides, it is an evident necessity to develop the concept of multimodel when at least two different membership functions for the same fuzzy set for one or more variables of the input are needed.

*Keywords:* Function approximator, membership functions, fuzzy modeling, fuzzy system. CSCC'99 Proc.pp.2041-2046

## 1 Introduction

Fuzzy systems offer a great flexibility for their application in different kinds of problems. For example, with a fuzzy model we have the possibility to take into account expert knowledge, but at the same time we can adapt the parameters of the model such that the model can be adapted in order to approximate a given function with a great degree of precision. The goal of this work is to show how we can improve the approximation capability of fuzzy models when we make better usage of membership functions of the input variables fuzzy sets. This way, for a given fuzzy model, an analytical function between the inputs and the output is found taking into account the relation that can exist between fuzzy sets and the fuzzy rules of the model. The analytical function is composed of the mathematical function used for membership functions.

There is some research where the authors demonstrate the influence of membership functions in the approximation capability of fuzzy models [1, 3-8, 10, 12, 13, 15]. However, in most of the works found in the literature, linear membership functions are used, and in many applications linear approximation may need a great quantity of fuzzy rules, a situation that could be impractical [2]. From a well defined inputs- output behavior, it is easier to select membership functions for fuzzy sets, such that the use of the appropriate shape of membership functions give us the possibility to reduce the fuzzy model size compared with triangular membership functions that give a linear approximation.

One of the main difficulties for fuzzy model construction consists in the definitions of the fuzzy sets for inputs and output variables. The problem is where to place the kernel of each fuzzy set and how to define the shape of each membership function. In [4-6] the authors show that for a good approximation with fuzzy models, the kernels of fuzzy sets must be placed in the maxima and minima of the function. In [5, 6] it is shown that the best way to approach a given function is to use a membership function that reproduces exactly the approximated function (see [6] for an example with two inputs). However, the construction of the fuzzy model with these membership functions is more difficult than that of the identification of the function that we want to approximate.

In [7] some results are given concerning the capability of the approximation of a fuzzy model when different kinds of membership functions are used. For all the shapes used, the function $sin(x)/x$, for the most part, proved to give the best results. The problem with these results is that the correlation between the function used for the fuzzy sets and the error of approximation is not clearly seen.

In other works, for example [13, 16], the authors take into account the dynamics of the approximation by using linear functions and/or B-splines. This way, the approximation can be from linear one, for linear functions [1, 11, 13, 15, 16] to approximations of superior order for B-splines. The limitation of this procedure is that the same function is used for all fuzzy sets.

There is another approach where the relation among the rate of change of the membership functions of the inputs to the output is taken into account in order to determine the rate of change of the output with respect to the inputs [8, 10]. Here, we are able to determine the influence near the kernels of fuzzy sets. However, we can not determine the output directly from the membership functions.

In order to get a good approximation, we make approximations by parties; knowing that fuzzy logic allows us to do it in a natural way. From this, it is easy to see that all fuzzy sets of the model are not restricted to a unique type of membership function, and when we need at least two membership functions for the same interval for one or more input variables, it is better to use the multimodel concept. This makes the interpretation of the fuzzy model easier and offers the possibility to get a good approximation in every party in which the model was broken. So, we also discussed the problem of modeling more than one maximum (or minimum).

The paper is organized as follows: in the next section we present the structure of the fuzzy model to be used in this work. Section three presents the analysis of the model in order to deduce the output as a mathematical function of the inputs and where the analytical functions for fuzzy sets construction appears explicitly. Finally, the fourth section presents some conclusions.

## 2 The fuzzy model

Consider the fuzzy model with T-norm product for the implication, $T(a,b)=a*b$, the Lukasiewicz T-conorm, $S(a,b)=min(a+b,1)$, for the aggregation and center of gravity method for defuzzification, $y=\sum_{k=1,\,n_y}\mu_{B'_k}(y)\cdot b_k \Big/ \sum_{k=1,\,n_y}\mu_{B'_k}(y)$.

There exists the possibility to use several fuzzy operators for the fuzzy model. However, for the operators that introduce the same non-linear behavior in the model, it does not seem very interesting because the non-linearity can not be controlled directly. This way, the non-linearity given by the operators in approximation problems is more of a disadvantage. This explains, in a certain measure, why Bounded sum-Product operators are chosen for applications where error is important in approximation. Besides, these operators give the best results for the approximation and allow a simpler analysis of the model. For example, in [13, 16] the authors compare the results when the minimum is used at the place of the product operator.

The defuzzification, as a part of the fuzzy model, can influence the quality of approximation. However, in this paper the defuzzification is used only as an interface between the fuzzy domain and the numeric domain [9].

Now, if input values are precise, the model can be analyzed by parties [11, 14] (see Fig. 1). For the model considered, take the index of the conclusions as the sum of the index of the fuzzy sets of input variables, as $k=i_1+i_2+\cdots+i_N$, and if the output fuzzy sets $B_k$ are in order, such as $B_0<B_1<\cdots<B_k<\cdots$ (or $B_0>B_1>\cdots>B_k>\cdots$), and uniformly distributed, then the relation among the output variable and the input variables are linear (see for example [1]).

When the index of the output fuzzy sets are given as the sum of the input fuzzy sets, the necessary fuzzy sets for the output can be calculated as $n_y=(1-N)+\sum_{j=1,\,N} n_j$, where $N$ is the number of inputs, and $n_y$ and $n_j$, $j=1, 2, \ldots, N$, are the number of fuzzy sets for the output and the inputs respectively.

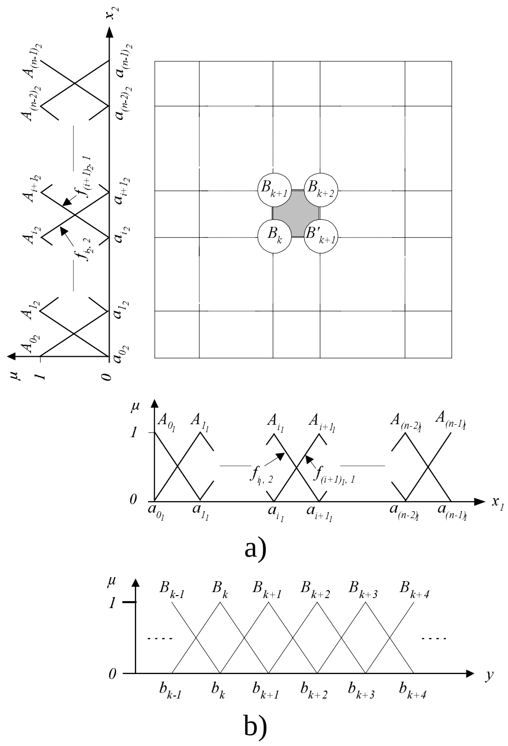


Fig. 1: Sections where the fuzzy rules can be activated at the same time.

## 3 Function approximation with the fuzzy model

In order to simplify the analysis, take into account only a section of the model as shown in Fig. 1. This model has two inputs, $x_1$ and $x_2$, and an output, $y$. The breakdown of the model is in accordance with the rules that can be activated at the same time [11, 14]. So, for $N$ input variables, there are $\prod_{j=1,N}(n_j-1)$ independent sections [14]. If we only take one section, the number of necessary output fuzzy sets, calculated with $n_y=(1-N)+\sum_{j=1,N} n_j$, is $n_y=N+1$.

For the model to be analyzed, the input variables, $x_1$ and $x_2$, are defined in $\lfloor a_{0_1}, a_{n_1-1} \rfloor$ and $\lfloor a_{0_2}, a_{n_2-1} \rfloor$ respectively. However, we take only the intervals $\lfloor a_{i_1}, a_{i_1+1})$ and $\lfloor a_{i_2}, a_{i_2+1})$ for the analysis in this paper, as it is shown in Fig. 1. In this way, we have 4 rules activated:

$$
\begin{aligned}
&\textit{If } x_1 \textit{ is } A_{i_1} \textit{ and } x_2 \textit{ is } A_{i_2} \textit{ Then } y \textit{ is } B_k\\
&\textit{If } x_1 \textit{ is } A_{i_1} \textit{ and } x_2 \textit{ is } A_{i_2+1} \textit{ Then } y \textit{ is } B_{k+1}\\
&\textit{If } x_1 \textit{ is } A_{i_1+1} \textit{ and } x_2 \textit{ is } A_{i_2} \textit{ Then } y \textit{ is } B'_{k+1}\\
&\textit{If } x_1 \textit{ is } A_{i_1+1} \textit{ and } x_2 \textit{ is } A_{i_2+1} \textit{ Then } y \textit{ is } B_{k+2}
\end{aligned}
\tag{1}
$$

In the most general case, the conclusions for fuzzy rules are all different. So, we consider this case at first and then we consider the index of the conclusions as the sum of the index of the inputs (we have several rules with the same conclusion). In the rules stated previously, that means $B_{k+1}=B'_{k+1}$. We use this condition when we generalize the results of the model with $N$ inputs.

The fuzzy sets $A_{i_j}$, $i=1,n_j$, for the inputs $x_j$, $j=1,2$, are defined as increasing, $f_{i_j,1}$, and decreasing monotonous functions, $f_{i_j,2}$, and they must be bounded.

$$
A_{0_c}(x_c)=\begin{cases} f_{0_c,2}(x_c) & a_{0_c}\le x_c<a_{1_c}\\ 0 & a_{1_c}\le x_c\end{cases}
$$

$$
A_{i_c}(x_c)=\begin{cases} f_{i_c,1}(x_c) & a_{i_c-1}\le x_c<a_{i_c}\\ f_{i_c,2}(x_c) & a_{i_c}\le x_c<a_{i_c+1}\\ 0 & a_{i_c-1}>x_c\ge a_{i_c+1}\end{cases}
\tag{2}
$$

$$
A_{n_c}(x_c)=\begin{cases} f_{n_c,1}(x_c) & a_{n_c-1}\le x_c\le a_{n_c}\\ 0 & x_c<a_{n_c-1}\end{cases}
$$

where $f_{i_c,d}:X_c\rightarrow[0,1]$, $c=1,2$, $d=1,2$, and $a_{i-1}<a_i<a_{i+1}$, $i=0,\dots,(n-1)$. Besides, the condition of strict fuzzy partition, $\sum_{i=1,n}\mu_{A_i}(x)=1,\ \forall x\in X$, is imposed for all the variables. For the fuzzy sets of the output $B_k$, we only use triangular membership functions. Furthermore, we suppose the kernels of the fuzzy set $B_{k+1}$ defined in the interval $[b_k, b_{k+2}]$. This allows for $b_{k+1}$ and $b'_{k+1}$ to be in this interval and then the fuzzy sets are in order. The order can be increasing or decreasing, it depends if we model a maximum or a minimum. The functions $f_{i_1+1,1}$ and $f_{i_2+1,1}$ can

also be defined as $f_{i_c+1,1}(x_c)=1-f_{i_c,2}(x_c)$, $\forall x_c \in [a_{i_c}, a_{i_c+1})$, $c = 1, 2$.

For the set of rules (1), the output corresponds to

$$y=\frac{N_0}{D_0}=\frac{(A_{i_1}(x_1)A_{i_2}(x_2))b_k+(A_{i_1}(x_1)A_{i_2+1}(x_2))b_{k+1}+}{A_{i_1}(x_1)A_{i_2}(x_2)+A_{i_1}(x_1)A_{i_2+1}(x_2)+}$$
$$\frac{(A_{i_1+1}(x_1)A_{i_2}(x_2))b'_{k+1}+(A_{i_1+1}(x_1)A_{i_2+1}(x_2))b_{k+2}}{A_{i_1+1}(x_1)A_{i_2}(x_2)+A_{i_1+1}(x_1)A_{i_2+1}(x_2)}$$

but, as we have a strict fuzzy partition, $D_0 = 1$, then the output is $y = N_0$.

Now, consider the relations $b_{k+1} = \beta_1 b_k + (1-\beta_1)b_{k+2}$ and $b'_{k+1} = \beta_2 b_k + (1-\beta_2)b_{k+2}$, with $\beta_1, \beta_2 \in [0, 1]$. With these relations and taking into account the hypothesis of the fuzzy strict partition, the output can be written as

$$y = b_k + (b_{k+2}-b_k)[(1-\beta_2)A_{i_1+1}(x_1)+(1-\beta_1)\cdot A_{i_2+1}(x_2)+(\beta_1+\beta_2-1)A_{i_1+1}(x_1)A_{i_2+1}(x_2)] \quad (3)$$

If we take $b_{k+1} = b'_{k+1}$, $\beta_2 = \beta_1$, the output is

$$y = b_k + (b_{k+2}-b_k)[(1-\beta_1)(A_{i_1+1}(x_1)+A_{i_2+1}(x_2))+(2\beta_1-1)A_{i_1+1}(x_1)A_{i_2+1}(x_2)] \quad (4)$$

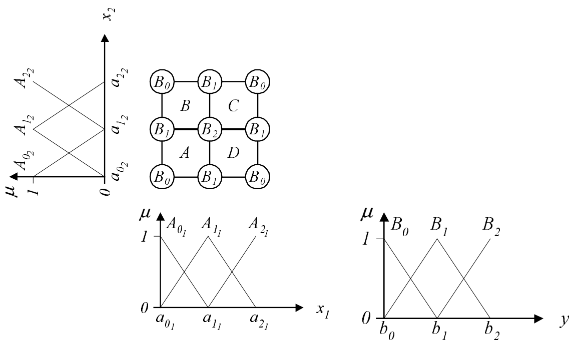
Notice that the main difference among (3) and (4) is the number of coefficients of each term. In (4) there is a coefficient for several terms, in (3) there is a coefficient for each term.

We can simplify (4) with the choice of a correct value for the kernel of $B_1$. For example, if we take $\beta_1 = 1/2$, or $b_{k+1} = (b_k + b_{k+2})/2$, we get the additive part of the equation. That is $y = b_k + ((b_{k+2}-b_k)/2)[A_{i_1+1}(x_1)+A_{i_2+1}(x_2)]$. From (2), we can write

$$y = b_k + \frac{(b_{k+2}-b_k)}{2}\left[f_{i_1+1,1}(x_1)+f_{i_2+1,1}(x_2)\right]$$

This equation shows how the output is a function of the inputs directly defined by the membership functions of the inputs fuzzy sets.

*Example:* Consider the modeling of a maximum of a function. This corresponds to $B_2$ in Fig. 2.a. The fuzzy sets of the inputs and the output are given in Fig. 2.a and Fig. 2.b. The fuzzy rules are also given in Fig. 2.a. For example, the fuzzy rule for the maximum is *If* $x_1$ *is* $A_{1_1}$ *and* $x_2$ *is* $A_{1_2}$ *Then* $y$ *is* $B_2$.



a) b)

Fig. 2: The fuzzy model for the maximum $B_2$.

In order to analyze the model consider four sections: $A$, $[a_{0_1}, a_{1_1})\times[a_{0_2}, a_{1_2})$, $B$, $[a_{0_1}, a_{1_1})\times[a_{1_2}, a_{2_2}]$, $C$, $[a_{1_1}, a_{2_1}]\times[a_{1_2}, a_{2_2}]$, and $D$, $[a_{1_1}, a_{2_1}]\times[a_{0_2}, a_{1_2})$ (see Fig. 2.a). If we take $b_1 = (b_2 + b_0)/2$ of $B_1$, the output for $A$, $B$, $C$, and $D$ is $y = b_0 + ((b_2-b_0)/2)[A_{1_1}(x_1)+A_{1_2}(x_2)]$.

Now, the fuzzy sets for the inputs $x_1$ and $x_2$ are

$$A_0(x)=\begin{cases}1-f_1(x) & 1\le x<2\\ 0 & 2\le x\end{cases}$$

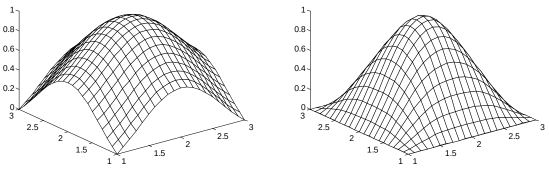
$$A_1(x)=f_1(x) \quad 1\le x\le 3$$

$$A_2(x)=\begin{cases}1-f_1(x) & 2\le x\le 3\\ 0 & x<2\end{cases}$$

with $f_1(x)=sin_1(\pi(x-2))/\pi(x-2)$. In this case, we have for the output

$$y=b_0+\left(\frac{b_2-b_0}{2}\right)\left[\frac{sin_{1_1}(\pi(x_1-2))}{\pi(x_1-2)}+\frac{sin_{1_2}(\pi(x_2-2))}{\pi(x_2-2)}\right]$$

that is defined by the membership functions of the inputs. The Fig. 3.a shows the results of simulation of the model.



a) b)

Fig. 3: Simulation of the fuzzy model with membership functions $sin(x)/x$ for the inputs.

Now, if we take $\beta_1 = 1$ in (4), that means $b_{k+1} = b_k$, the output is $y = b_k + (b_{k+2} - b_k)A_{i_1+1}(x_1)A_{i_2+1}(x_2)$, or as an explicit function of membership functions

$$y = b_k + (b_{k+2} - b_k)[f_{i_1+1,\,1}(x_1)\cdot f_{i_2+1,\,1}(x_2)]$$

So, with the membership functions defined by $sin(x)/x$, the output is

$$y = b_0 + (b_2 - b_0)\left[\frac{sin_{1_1}(\pi(x_1-2))}{\pi(x_1-2)}\cdot\frac{sin_{1_2}(\pi(x_2-2))}{\pi(x_2-2)}\right]$$

The results of simulation of the model are showed in Fig. 3.b.

### 3.1 Fuzzy model with N inputs

As in the previous case, the output of the fuzzy model is a multilinear function when we have $N$ inputs $(N \geq 2)$. So, if we use the right value for the kernels of the output fuzzy sets, we can simplify the multilinear function. For example, consider the input variables $x_1, x_2, \ldots, x_N$, defined in $X_1, X_2, \ldots, X_N$, and the output *y*, defined in *Y*. For the fuzzy sets of $x_j$, $j = 1, N$, we use the notation $A_{i_j}$, $i_j = 0, n_j - 1$, and for $y$ the notation is $B_k$, $k = 0, n_y - 1$, with kernels $b_k$. For each variable, the fuzzy sets form a strict fuzzy partition.

For the analysis take only a part of the model, as shows Fig. 1 for two variables. For this section of the model we need $n_y = N + 1$ fuzzy sets of the output. Furthermore, if the fuzzy set $B_k$ of the output for each rule is defined in function of the input fuzzy sets $A_{i_j}$ and $A_{i_j+1}$ for each variable $x_j$, that is $k = i_1 + i_2 + \cdots + i_N$, the fuzzy rules can be written as

$$\text{If } x_1 \text{ is } A_{i_1} \text{ and } x_2 \text{ is } A_{i_2} \text{ and ... and } x_N \text{ is } A_{i_N} \text{ Then } y \text{ is } B_k$$

$$\vdots$$

$$\text{If } x_1 \text{ is } A_{i_1+1} \text{ and } x_2 \text{ is } A_{i_2+1} \text{ and ... and } x_N \text{ is } A_{i_N+1} \text{ Then } y \text{ is } B_{k+N}$$

Now, if the kernels of $B_{k+a}$, $a = 1, 2, \ldots, N-1$, must be between those of $B_k$ and $B_{k+N}$, that is, in the interval $[b_k, b_{k+N}]$, the kernels can be calculated as $b_{k+a} = \beta_a b_k + (1-\beta_a)b_{k+N}$, $a = 1, 2, \ldots, N-1$, $\beta_a \in [0,1]$.

The relation among the inputs and the output is

$$y = \sum_{k_1, k_2, \ldots, k_N = \{0,1\}} c_{(k_1+k_2+\cdots+k_N)} A^{k_1}_{i_1+1}(x_1)\cdots A^{k_N}_{i_N+1}(x_N)$$

where $x = \{x_1, x_2, \ldots, x_N\}$, and $c_{(k_1+k_2+\cdots+k_N)}$ are the coefficients of the function.

In order to simplify this relation, the kernels $b_{k+a}$ of the output fuzzy sets must be uniformly distributed in the interval $[b_k, b_{k+N}]$. That is simple if we take the kernels $b_{k+a} = b_k + (a/N)(b_{k+N} - b_k)$, $a = 1, 2, ..., N-1$. Then, the output is

$$y = b_k + \frac{(b_{k+N} - b_k)}{N}\sum_j A_{i_j}(x_j)$$

If the kernels of the output fuzzy sets are $b_{k+a} = b_k$, for $a = 1, 2, ..., N-1$, the output is

$$y = b_k + (b_{k+N} - b_k)\prod_j A_{i_j}(x_j)$$

Here, the most important is that these expressions can be written such that the input membership functions appear explicitly (see (2)). However, if we have more than one maximum or minimum or both to model, but the shape for each one is different, we have to separate each maximum or minimum and model each one with a fuzzy model. See the next figure for example where each maximum and/or minimum is considered to be modeled by a model, $M_1$ and $M_2$, and not as two sections of one model.

When we only use a model, we need to choose a membership function of both needed for the model; or a compromise of them. This is not the case for the multimodel which in reality offers a great flexibility for using several membership functions for the same fuzzy set but where we have the possibility to make a good local approximation for each section considered.

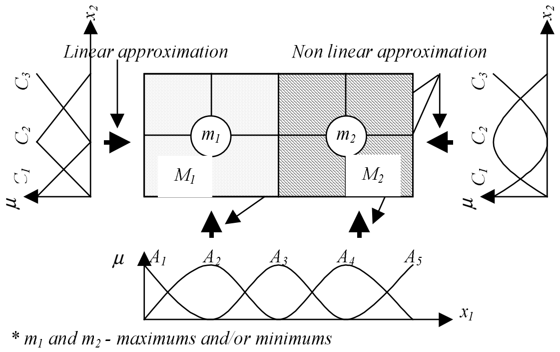



Fig. 4: Two models for linear and non linear approximation of maximums and/or minimums.

## 4 Conclusions

The interest of the results is that we can utilize an analytical expression, where the influence of the functions used for the fuzzy sets is stated clearly with respect to the output. This way, the choice of the membership functions for each input fuzzy set depends on local behavior.

This allows us to use the known facts about the behavior of the system to be modeled into the fuzzy model, because we can translate this behavior by the membership functions of the inputs. Besides, the better approximation of the behavior with the fuzzy sets allows using fewer elements in the model, so we can reduce the size of the model with good approximation results.

Finally, when we have several maximums or minimums, or both, the concept of multimodel, each model for a section of the initial model, can be used in order to define different membership functions that allows for a good approximation. Nevertheless, the problem with the transition between the models, if the output corresponds to an additional function, still remains. But this is not a problem when the output is a product of the inputs.

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