Liveness of Continuous Weighted Marked Graphs

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Abstract: - For modelling and analysis of production systems as batch or high throughput systems, Petri Nets are commonly used with a concept of flow. Furthermore, for performance evaluation and performance control, discrete Petri Nets (PN) with structures such that marked graphs (also called event graphs) have produced a great interest by their analysis facilities. These marked graphs correspond to a net structure where a place has exactly one input and one output transition and where weights on arcs are equal to one. Thus, they are powerful enough to model and analyse cyclic systems.

But they are several problems associated with the introduction of weighted larger than one. These weights may be introduced to reduce the size of the model. Consequently, weighted marked graphs permit to model in a compact structure, batch systems, assembly and disassembly systems, where transformations on product appear.

However, when the discrete PN contains many tokens, the number of reachable states explodes. One way to reduce the phenomena, is the use of continuous PN. In this continuous model [1,4,5], the marking of place is a non-negative real number, introducing the notion of product flow in tank for instance. Notion of time is represented on transition as maximal firing speed. One way to determine structural properties such that liveness (assuring the non deadlocked behaviour), is to construct the evolution graph. In this graph, a node represents the instantaneous firing speed vector. This vector is constant over a certain period, since evolutions of the markings are linear time functions.

Nevertheless, when the structure of the PN is a strongly connected marked graph, the liveness of a continuous PN can be established by the study of its directed circuits. But to our knowledge, no work has been done when weights are added on arcs.

The present paper extends previous works [3,5,10] for continuous weighted marked graphs. By the notion of loop gain, we establish the necessary and sufficient conditions for the liveness of a continuous circuit. For specific initial conditions on marking, the exact values of final instantaneous speeds are given.
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1 Introduction

Performance control or performance evaluation of batch processes or high throughput production systems pose difficult issues because their representation deals with continuous and discrete models. At the physical level, these processes are usually considered with a fluid point of view. This flow can be characterised by a real number expressing the amount of material.

Relative to Petri net (PN) theory, discrete PNs [9] have generated wide interest for the design and operation of manufacturing systems, while continuous PNs have found a great importance for the modelling and analysis of fluid processes [6]. An other advantage of continuous PNs is that every property of a discrete PN (DPN), deduced from the fundamental equation, can be transposed to a continuous PN (CPN). In particular, results for Place-invariants and Transition-invariants are similar for a DPN and for a CPN [1]. But it seams that concepts of liveness, boundedness and dead lock-freeness, are quite equivalent for continuous and discrete PN. Nevertheless to determine these basic structural properties without using simulation or evolution graphs, few works have been done.

By restricting the approach to structure of marked directed graphs, Commoner and al [3] prove that “a marking is live if and only if the token count of every directed circuit is positive” and that “a marking which is live remains live after firing”. In terms of Continuous Petri net, the former proposition means that “a marking is live if and only if every ordinary (elementary) circuit contains at least one place with a positive marking”. The latter proposition means, “every transition which is enabled at a certain time, stays enabled whatever the future evolution”. Consequently for a continuous non weighted marked graph, David and Alla [5] deduce conditions to reach the final steady state and give the exact values of the final instantaneous firing speeds.

Unfortunately, no works have been done for continuous weighted marked graphs and only few for discrete weighted marked graphs [10]. In this paper, we demonstrate the liveness of continuous weighted marked graph under loop gain conditions. This property on sampled structure allows to represent permanent behaviour of cyclic systems and characterises the steady state.

The primary section of this paper is dedicated to some recalls on continuous Petri nets. Section 3 illustrates, through of an example, the liveness of a continuous weighted circuit, gives definitions of gains and basic notations. Section 4 presents fundamental properties on continuous paths and defines conditions for the liveness of continuous circuits. In case of neutral circuits, exact values of final firing speeds are established. Followed by conclusions, last section generalises the conditions of liveness for continuous weighted marked graphs.

2 Background on Continuous Petri nets

We assume that the reader is familiar with the Petri nets paradigm (for more details, see [5, 8]). Since all continuous Petri nets considered in this paper have the characteristic of constant speeds, this adjective may be implicit.

Continuous Petri nets [4] allow the modelling of some continuous systems [6]. The marking of places is a real number and the firing of transitions is a piecewise constant function.

Definition 1: A ***timed Continuous Petri net*** (CPN) is a sextuple $C = (P, T, Pre, Post, M_o, F)$ such that:

P is a finite set of places and T is a finite set of transitions,

Pre, $Post$: $PxT \rightarrow Â^+$ are input and output incidence mappings which associate to each arc a non negative real number,

M_o is the initial marking which takes its value in the set of non-negative real numbers,

F : T ® $Â^+$, the application which associates at each transition a non negative real number, named maximal firing speed F_j (also called maximal firing flow).

Notations:

- $m_i(t)$ is the marking of place P_i, $m_i(t)=m(P_i)(t)$, and $M(t)$ is the marking of the Petri net (vector of markings), at time t.
- m_i^0 is the initial marking of the place P_i (at the initial date t_0), $m_i^0 = m_i(t_0)$, and M_0 is the initial marking of the CPN.
- $°T_j$ and $T_j°$ (resp. $°P_i$ and $P_i°$) are sets of the input and output places (resp. transitions) of the transition T_j (resp. place P_i)

Before the presentation of enabling rules of a CPN, we first introduce the specific concept of a *fed continuous place*, which does not exist in discrete PN. An empty place can be supplied by an input transition, which is enabled. Thus, as a flow can pass through an unmarked continuous place, this place can deliver a flow to its output transitions.

Consequently, *transition* T_j *is enabled* at time t if and only if all its input places P_i satisfy at least one of the following conditions :

- $m_i(t) > 0$.
- P_i is fed.

If all input places of T_j satisfy the first condition, i.e. they have a not null marking, T_j is called *strongly enabled.*

If some of input places are fed, T_j is called *weakly enabled*.

Finally, transition T_j is *not enabled* if one of its input empty places is not fed.

To represent the dynamic behaviour of a CPN, an instantaneous firing speed $j_j(t)$ is associated with each transition T_j. This instantaneous firing speed is a piecewise constant function. Transitions are fired with this real speed which must be lower than the maximal one. So at any time, $j_j(t)$ £ Φ_j.

At time t :

- If T_j is a strongly enabled transition then its instantaneous firing speed is equal to its maximal firing speed.

$$\varphi_j(t) = \Phi_j \quad (1)$$

- If Tj is a weakly enabled transition then its instantaneous firing speed is given by:

$$\varphi_j(t) = \min [\Phi_j, \min_{i/ P_i \in °T_j} (B_i(t) + \varphi_j(t))] \quad (2)$$

where $B_i(t)$ is the *dynamic balance* of place P_i. More precisely, $B_i(t)$ represents the variation (increasing or decreasing) of the marking $m_i(t)$ when place P_i has an input or/and output flow.

The dynamic balance of place P_i is equal to :

$$B_i(t) = \sum_{k=1}^{n} Post(P_i,T_k)\cdot j_k(t) - \sum_{k=1}^{n} Pre(P_i,T_k)\cdot j_k(t) \quad (3)$$

Remarks:

1) $B_i(t) > 0 \Rightarrow$ the marking of place P_i increases
2) $B_i(t) < 0 \Rightarrow$ the marking of place P_i decreases
$B_i(t)$ can not be negative if m_i (t) = 0. This condition guaranties the marking to be a non negative number.
3) $B_i(t) = 0 \Rightarrow$ the marking of place P_i is stable

B(t) denotes the dynamic balance vector composed of each dynamic balance $B_i(t)$ of places.

As a result, instantaneous firing speeds depend on both, the dynamic balance and the state of places (marked or empty). At a fixed date, a change of the speed vector will occur if there are strictly negative dynamic balances at this date, meaning that some markings decreases. So, when the marking of a place becomes null, an event occurs, the new dynamic balance *B(t)* is calculated and a new instantaneous firing speeds vector *j (t)* is determined. Consequently, in a continuous Petri net, if each place has a non negative dynamic balance, this characterised state is the *steady state* (called also permanent state). More details on algorithms for computing the instantaneous firing of transitions can be found in [4] but the dynamic principle will be illustrated by an example in the next section.

Thanks to Commoner and al. [3], it is well known that a *Petri net* is *live* if and only if every transition can ultimately occur from any reachable marking
But there are several important problems associated with the liveness and speeds determination of a continuous PN. To avoid conflict problems, we restrict our approach to PN structure as weighted marked graphs (also called weighted T-graphs or weighted event graphs). Continuous weighted marked graphs correspond to a Petri net where each place has exactly one input and one output transition.

For reasons of better handling, we first study the liveness of a continuous weighted elementary circuit (an elementary circuit contains each node (place or transition) at most one time). Lastly, established conditions will be generalised for continuous weighted marked graph, as strongly connected graph can be decomposed into elementary circuits.

3 Basic definitions on weighted circuits

This section begins with an intuitive and informal presentation of liveness, illustrated by an example. Next, we define basic notations and definitions that are going to be used and we study some properties on continuous paths.

3.1 Intuitive presentation

The CPN given in figure 1 models a manufacturing line is composed of four machines separated by intermediary buffers with infinite capacity.

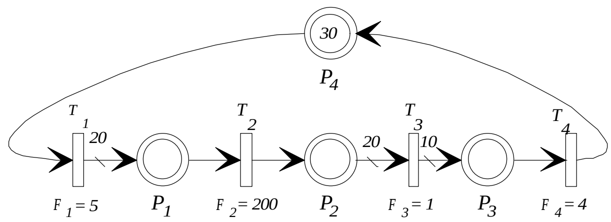

Fig. 1 : Continuous elementary circuit of a production system

Machine M1 (transition T_1) with a maximal rate of 5 batches per second, splits one batch to 20 identical components and puts them in buffer S1 (place P_1). Machine M2 (transition T_2), with a maximal rate of 200 components per second puts its outgoing parts in buffer S2. Machine M3 (illustrated by T_3) assembles 20 components from stock S2 in 10 batches. With a maximal throughput of 1 transformation per second, M3 puts then 10 batches in buffer S3 (place P_3) for each transformation. Finally, at a maximal speed of 4 batches per second, outgoing pieces of machine M4 (transition T_4) finish in buffer S4 (modelled by P_4). This last buffer is also the input buffer of M1. This manufacturing system is cyclic and contains initially 30 batches in buffer S4.

At the initial date t_0 = 0 second, buffer S4 is non empty so M1 operates at its maximal speed (5 batches/sec.). A flow of 20*5 = 100 components/sec. arrives through buffer S1 in front of machine M2, which has its maximal speed higher and equal to 200. So, M2 is not saturated and its instantaneous rate is 100 components/sec. Thus, buffer S1 stays empty. In the same time, a flow of 100 components/sec. fills buffer S2. Next, 20 components from S2 are grouped into 10 batches by machine M3. Since M3 has a maximal rate of 1 transformation/sec., it can not process a flow of 100/20 = 5 transformations/sec. So, machine M3 is saturated and buffer S2 is filling up. Machine M4 receives a flow equal to 10 batches/sec. Since its maximal rate is 4, M4 is saturated and buffer S3 is filling up too. Finally, buffer S4 receives an input flow equal to 4 while its output flow is equal to 5. So the number of batches in buffer S4 decreases. As shown on figure 2, at the initial date, the dynamic balance of P_4 is negative, while dynamic balances of P_1, P_2, and P3 are positive or null. Finally, the instantaneous firing speed vector is φ(t=0s)=(5,100,1,4).

After a delay of 30 seconds, buffer S4 is empty (marking of P_4 equals zero), and buffers S1, S2 and S3 contain respectively 0, 2400 and 180 parts. At this date t = 30sec., M3 and M4 are always saturated, transitions T_3 and T_4 are strongly enabled, thus their instantaneous firing speeds are, respectively, 1 and 4. Machine M1 will receive batches through buffer S4 (P_4 is fed), with an instantaneous rate imposed by M4 (lower than its maximal one). So instantaneous rate of M1 is equal to 4 batches/sec. As buffer S1 is empty (but place P_1 is fed), machine M2 has a throughput of 20 * 4 = 80 components/sec. Lastly, at date 30 second, the dynamic balances of P_4 and P_1 are null (see figure 2), while they are strictly positive for P_2 and P_3, and the instantaneous firing speed vector is equal to $\varphi(t=30)=(4,80,1,4)$.
Since this state is characterised only by positive or null dynamic balances, we reach the final steady state (no event will occur). So, the throughputs of machines remain constant (4,80,1,4), the number of parts in the buffers S2, S3 increases while in buffers S1, S4 the number of part remains null. The permanent behaviour is reached and the circuit is live.

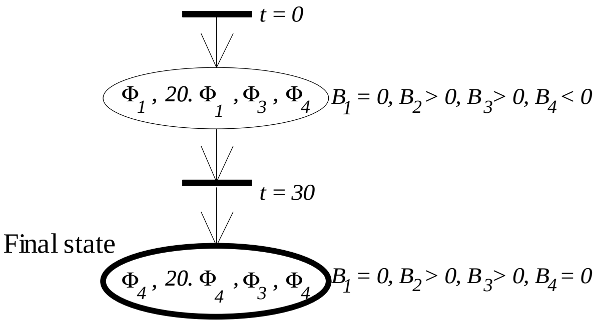

Fig 2 : Evolution graph for one circuit

This example illustrates that every continuous elementary circuit which contains at least one non empty place and a positive loop gain (see definition 3) is live and reaches the permanent steady state.

3.2 Basic notations and definitions

For an elementary circuit composed of n places and n transitions, $g = \langle T_1, P_1, T_2, P_2, \ldots, T_n, P_n \rangle$, we use the following notations:

- $\forall$ i, w_i = Post (P_i, T_i) is the weight of the input arc of the place P_i.
- $\forall$ i $\neq$ n, v_i = Pre (P_i, T_{i+1}) is the weight of the output arc of the place P_i.
- For i = n, v_n = Pre (P_n, T_1) is the weight of the output arc of the place P_n.

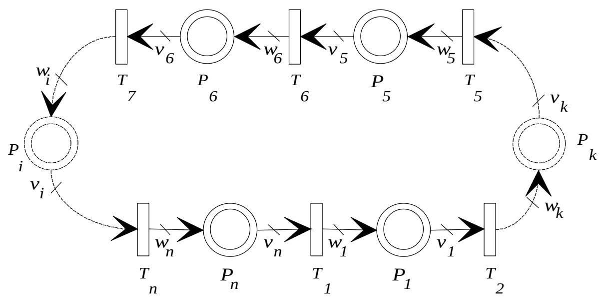

Fig. 3 Weighted elementary circuit.

Definition 2: We define the ***gain of a directed path*** from a transition T_k to a transition T_j by:

$$G_k^j = \prod_{i=k}^{j-1} \frac{w_i}{v_i} \tag{4}$$

❑

If the first index is greater than the second index (i.e. k > j-1), the index i goes from the first index k to n and continues from 1 to the second index j-1.
For example, the gain of the path from T_7 to the transition T_5 is given by :

$$G_7^5 = \prod_{i=7}^{4} \frac{w_i}{v_i} = \frac{w_7}{v_7} \ldots \frac{w_n}{v_n} \cdot \frac{w_1}{v_1} \ldots \frac{w_4}{v_4}$$

Remark: In the following, for all functions using index (such that *Min*, *P*), if the first index k is greater than the second index j, the index i goes from the first index k to n and continues from 1 to the second index j. Likewise, for an elementary circuit or a path, when we arrive to place P_n, we continue from transition T_1.

Definition 3: The ***gain G of an elementary circuit*** *g* (also called loop gain of *g*) is defined by:

$$G(g) = \prod_{i=1}^{n} \frac{wi}{vi} \tag{5}$$

❑

In [2,10], each elementary circuit can be classified in three cases, depending on the value of its gain:

1. *G(g) = 1,* the circuit is neutral.
2. *G(g) < 1,* the circuit is absorbing.
3. *G(g) > 1,* the circuit is generating.

4 Liveness of elementary continuous circuits

This part studies the steady state of a continuous elementary circuit with weights on arcs. Firstly, we prove some properties on weighted continuous paths and study the dynamic behaviour. Next, by using these properties, necessary and sufficient conditions for liveness of continuous circuits are defined. Finally, the theorem of liveness for neutral circuit (under marking assumptions) determines instantaneous firing speed expressions for the steady state.

4.1 Dynamic of continuous weighted paths

With the above notations, in a continuous weighted circuit, at time t, the dynamic balance of a place P_i (eq. 3), is now expressed as follow:

$$B_i(t) = -v_i.\varphi_{i+1}(t) + w_i.\varphi_i(t) \qquad (6)$$

For a weakly enabled transition T_j, its instantaneous firing speed at time t can be deduced from eq. (2):

$$\varphi_j(t) = \min\left(\Phi_j, -v_{j-1}.\varphi_j(t) + w_{j-1}.\varphi_{j-1}(t) + \varphi_j(t)\right).$$

If $\varphi_j(t)$ is equal to the second term,

$$\varphi_j(t) = -v_{j-1}.\varphi_j(t) + w_{j-1}.\varphi_{j-1}(t) + \varphi_j(t)$$

we can easily deduce: $\varphi_j(t) = \frac{w_{j-1}}{v_{j-1}}.\varphi_{j-1}(t)$.

By including the second possibility, $\varphi_j(t)=\Phi_j$, the instantaneous firing speed of a weakly enabled transition T_j is:

$$\varphi_j(t) = \min(\Phi_j, \frac{w_{j-1}}{v_{j-1}}.\varphi_{j-1}(t)) \qquad (7)$$

With these above equations, four properties are now established for a path. The first one gives the instantaneous firing speed of transitions, for a one-marked path (a path with only the first place marked). The property 2 provides, for a two-marked path (a path with the first place and the last place marked), a superior bound of instantaneous firing speeds and a condition to have a non positive dynamic balance for the last place. Property 3 extends some results of the second property to the case of a multi-marked path. Finally, in addition to property 1, property 4 analyses the dynamic behaviour, for a one-marked path.

Property 1: Let a continuous path r = <P_i, T_{i+1}, P_{i+1}, …, P_{q-1}, T_q>, such that at time t, place P_i is the unique marked place while the other places P_k, k = i+1, …, q-1, are empty. Every transition T_j ,for j = i+1 to q, is enabled and has its instantaneous firing speed given by:

for j = i+1, $\varphi_{i+1}(t) = \Phi_{i+1}$ (8)

for j = i+2, …, q,

$$\varphi_j(t) = \min\left(\Phi_j, \min_{k=i+1}^{j-1}\left(\Phi_k.\prod_{p=k}^{j-1}\frac{w_p}{v_p}\right)\right) \qquad (9)$$

Expressed with the gain of a path (eq. 6), the last equation is transformed as follow:

$$\varphi_j(t) = \min\left(\Phi_j, \min_{k=i+1}^{j-1}\left(\Phi_k.G_k^j\right)\right) \qquad (10)$$

❑

Proof: Let us consider a path r = <P_i, T_{i+1}, P_{i+1}, …, P_{q-1}, T_q > such that only the place P_i is marked (all the others are supposed empty), at time t, i.e. :

$m_i(t) = \varepsilon > 0$ *and* $\forall P_{j \neq i} \in r$, $m_j(t) = 0$.

We are going to prove the formula (9) by recurrence and show that each transition is enabled at time t.

- Transition T_{i+1} is strongly enabled since its single input place P_i is marked. From equation (1), its instantaneous firing speed is given by :

  $\varphi_{i+1}(t) = \Phi_{i+1}$. (11)

- Transition T_{i+2} is weakly enabled since its single input place P_{i+1} (= $°T_{i+2}$) is empty and fed. As the instantaneous firing speed of transition T_{i+2} is done by equation (7), by replacing the value of the instantaneous firing speed of T_{i+1} (eq. 11), the instantaneous firing speed of T_{i+2} is:

  $$\varphi_{i+2}(t) = \min(\Phi_{i+2}, \frac{w_{i+1}}{v_{i+1}}.\Phi_{i+1}) \qquad (12)$$

- Transition T_{i+3} is also weakly enabled.
  By replacing the instantaneous speed of T_{i+2} in equation 7 adapted to T_{i+3}, we deduce:

  $$\varphi_{i+3}(t) = \min(\Phi_{i+3}, \frac{w_{i+2}}{v_{i+2}}.\Phi_{i+2}, \frac{w_{i+2}}{v_{i+2}}\frac{w_{i+1}}{v_{i+1}}.\Phi_{i+1}) \qquad (13)$$

- Now, we assume that the transition T_j. is weakly enabled that its instantaneous firing speed is given by formula (9).
  Let us prove that this formula is always verified for T_{j+1}.

- Transition T_{j+1} is weakly enabled:
  Hence,

  $$\varphi_{j+1}(t) = \min(\Phi_{j+1}, \frac{w_j}{v_j}.\varphi_j(t)),$$

  by including equation (9),

  $$\varphi_{j+1}(t) = \min\left[\Phi_{j+1}, \frac{w_j}{v_j}.\min\left(\Phi_j, \min_{k=i+1}^{j-1}\left(\Phi_k.\prod_{p=k}^{j-1}\frac{w_p}{v_p}\right)\right)\right]$$

  in other terms,

  $$\varphi_{j+1}(t) = \min\left[\Phi_{j+1}, \frac{w_j}{v_j}\Phi_j, \frac{w_j}{v_j}\min_{k=i+1}^{j-1}\left(\Phi_k.\prod_{p=k}^{j-1}\frac{w_p}{v_p}\right)\right]$$

  or

  $$\varphi_{j+1}(t) = \min\left[\Phi_{j+1}, \frac{w_j}{v_j}\Phi_j, \min_{k=i+1}^{j-1}\left(\Phi_k.\prod_{p=k}^{j-1}\frac{w_p}{v_p}\frac{w_j}{v_j}\right)\right]$$

  Finally we obtain,

  $$\varphi_{j+1}(t) = \min\left[\Phi_{j+1}, \min_{k=i+1}^{j}\left(\Phi_k.\prod_{p=k}^{j}\frac{w_p}{v_p}\right)\right] \qquad (14)$$

  As all transitions from T_{j+1} are weakly enabled, we can do this reasoning until last transition T_q of the path.

❑

Property 2: Let a continuous path r = <P_n, T_1,…, P_{j-1}, T_j, P_j, T_{j+1}, >, such that at time t places P_n and P_j are the unique marked places while the other places P_k, k = 1, …, j-1, are empty.

- For transitions T_1 and T_{j+1} : $j_1(t)=\Phi_1$ and $j_{j+1}(t)=\Phi_{j+1}$.
  For each transition T_s (weakly enabled), s = 2 to j, we have:
  $$j_s(t) \le \Phi_1.G_1^s \qquad (15)$$
  Furthermore, if $B_k(t) = 0$ for k = 1, …, s-1,
  $$j_s(t) = \Phi_1.G_1^s \qquad (16)$$
- For the place P_j,
  if $\Phi_{j+1} \ge \Phi_1.G_1^{j+1} \Rightarrow B_j(t) \le 0$ (17)
  if $B_j(t) > 0 \Rightarrow F_{j+1} < F_1.G_1^{j+1}$ (18)

❑

Proof: According to property 1, the instantaneous firing speed of a weakly enabled transition T_s, for s=2 to j, equals (see eq. 10):

$$j_s(t) = \min\left(\Phi_s, \min_{k=1}^{s-1}\left(\Phi_k.G_k^s\right)\right),$$

Thus, $j_s(t) \le \Phi_1.G_1^s$, so equation (15) is verified.

Moreover, if every dynamic balance from P_1 to P_{s-1} is null: $B_k(t) = w_k.j_k(t) - v_k.j_{k+1}(t) = 0$ (k = 1 to s-1**),** the instantaneous firing speed of the input transitions (weakly enabled) is: $j_{k+1}(t) = j_k(t)\dfrac{w_k}{v_k}$

Hence, by recurrence, we deduce:

$$j_s(t) = j_{s-1}(t)\frac{w_{s-1}}{v_{s-1}} = ... = j_1(t)\frac{w_1}{v_1}...\frac{w_{s-1}}{v_{s-1}} = j_1(t).G_1^{s-1}$$

As transition T_1 is strongly enabled, c, we obtain equation (16): $j_s(t) = \Phi_1.G_1^s$

Finally, the output transition of the last marked place P_j of r, is strongly enabled, so: $j_{j+1}(t) = \Phi_{j+1}$.

From equation (15) applied to T_j, $j_j(t) \le \Phi_1.G_1^j$,

then $B_j(t) = w_j.j_j(t) - v_j.j_{j+1}(t) \le w_j.\Phi_1 G_1^j - v_j.\Phi_{j+1}$

* if $\Phi_{j+1} \ge \Phi_1 G_1^j.\dfrac{w_j}{v_j}$ (or $\Phi_{j+1} \ge \Phi_1 G_1^{j+1}$) then $B_j(t) \le 0$

* if $B_j(t)>0$ then $F_{j+1} < F_1.G_1^{j+1}$

❑

Property 3: Let a continuous path r = <P_n, T_1,…, P_{j-1}, T_j, P_j> such that at time t, places P_n and P_j are marked, and there are some marked places P_i, i = 1, …, j-1, with a strictly dynamic balance, $B_i(t)>0$,

Then, for each transition T_s, for s = 2 to j, its instantaneous firing speed is bounded by:

$$j_s(t) \le \Phi_1.G_1^s \qquad (19)$$

if $B_s(t) > 0 \Rightarrow F_s < F_1.G_1^s$

❑

Proof: We suppose that path r contains, at time t, q marked places. Let c(i) the i[th] marked place in the path r with c(1) = n and c(q) = j.

We can decompose path r in (q-1) paths r_i = <$P_{c(i)}$, $T_{c(i)+1}$,…, $P_{c(i+1)-1}$, $T_{c(i+1)}$> which have only their first place marked with strictly positive dynamic balances (other places are empty).

- From property 2 (eq. (15) and (18)), applied to path $r_1 \cup \{P_{c(2)}\}$, we have:
  $j_1(t) = \Phi_1$
  $j_s(t) \le \Phi_1.G_1^s$ for s = c(1)+2, …, c(2),
  and $j_{c(2)+1}(t) = F_{c(2)+1} < F_1.G_1^{c(2)+1}$
- For the path $r_2 \cup \{P_{c(3)}\}$, property 2 implies:
  $j_{c(3)+1}(t) = F_{c(3)+1} < F_1.G_1^{c(3)+1}$, and
  $j_s(t) \le \Phi_{c(2)+1}.G_{c(2)+1}^s$ for s = c(2)+2 to c(3).
  By including equation (20)
  $$j_s(t) \le \Phi_{c(2)+1}.G_{c(2)+1}^s < \Phi_1.G_1^{c(2)+1}.G_{c(2)+1}^s = \Phi_1.G_1^s$$

And so on for all paths r_i.

By the same reasoning for every transition T_s of the path r at time t, we obtain the following condition:

$$j_s(t) \le F_1.G_1^s$$

❑

Property 4: Let a continuous path r = <P_i, T_{i+1}, P_{i+1}, …, P_{q-1}, T_q>, such that at time t_0, place P_i is the unique marked place while the other places P_k, k = i+1, …, q-1, are empty.

- If at time t_0 the dynamic balance of an empty place P_k is null, it will be stay null whatever the future evolution.
  At time t_0, $B_k(t_0) = 0$ then at time $t > t_0$, $B_k(t) = 0$.
- If at time t_0 the dynamic balance of an empty place P_k, is positive, whatever the future evolution, $F_{k+1} < F_1.G_1^{k+1}$ will be verified for transition T_{k+1}.
  At time t_0, $B_k(t_0) > 0$ then at time $t > t_0$, $F_{k+1} < F_1.G_1^{k+1}$ whatever the future value of $B_k(t)$.
- If at time t_0 the dynamic balance of two empty places P_k and P_q (P_k is upstream to P_q) are positive, and the dynamic balance of all the places between them are null, then dynamic balance of P_q remains positive as long as P_k is marked.
  At time t_0, $B_k(t_0) > 0$ and $B_q(t_0) > 0$, if at time $t > t_0$ we have $m_k(t) > 0$ then $B_q(t) > 0$.

❑

Proof:

- Firstly, we are going to prove that for all empty places, between T_{i+1} and T_q, which have a null dynamic balance at time t_0, their dynamic balance stays null, whatever future time t> t_0.

At time t_0, the place P_{i+1} is empty ($B_{i+1}(t_0)$=0), and T_{i+1} is strongly enabled $j_{i+1}(t)=F_{i+1}$. According to equation (7) for transition T_{i+2}:

$$j_{i+2}(t_0)=min(F_{i+2},\frac{w_{i+1}}{v_{i+1}}.F_{i+1})$$

As $B_{i+1}(t_0)$ = 0, from equation (6) we deduce that:

$$j_{i+2}(t_0)=\frac{w_{i+1}}{v_{i+1}}.\Phi_{i+1}\leq\Phi_{i+2}$$

Therefore, so long as place P_i is marked, for t> t_0, we have $j_{i+2}(t)=\frac{w_{i+1}}{v_{i+1}}.\Phi_{i+1}$

and $B_{i+1}(t)=w_{i+1}.\Phi_{i+1}-v_{i+1}.\frac{w_{i+1}}{v_{i+1}}\Phi_{i+1}=0$.

At time t_0, the place P_{i+2} is empty ($B_{i+2}(t_0)$=0), and T_{i+1} is strongly enabled $j_{i+1}(t)=\Phi_{i+1}$. According to equation (7) for transition T_{i+3}:

$$j_{i+3}(t_0)=\min(\Phi_{i+3},\frac{w_{i+2}}{v_{i+2}}.j_{i+2}(t_0)).$$

As $B_{i+2}(t_0)$ = 0, from equation (6) we deduce that:

$$j_{i+3}(t_0)=\frac{w_{i+2}}{v_{i+2}}.j_{i+2}(t_0)=\frac{w_{i+2}}{v_{i+2}}\frac{w_{i+1}}{v_{i+1}}.\Phi_{i+1}\leq\Phi_{i+3}$$

Therefore, so far as place P_i is marked, for t > t_0, we have $j_{i+3}(t)=\frac{w_{i+2}}{v_{i+2}}\frac{w_{i+1}}{v_{i+1}}.\Phi_{i+1}=G_{i+1}^{i+3}.\Phi_{i+1}\leq\Phi_{i+3}$

and

$$B_{i+2}(t)=w_{i+2}.j_{i+2}(t)-v_{i+2}.G_{i+1}^{i+3}\Phi_{i+1}$$

$$=w_{i+2}.G_{i+1}^{i+2}\Phi_{i+1}-v_{i+2}.G_{i+1}^{i+3}\Phi_{i+1}=0$$

By the same reasoning if the dynamic balance of an empty place P_k (between P_i and P_q) is null, it will be stay null whatever the future evolution.

- To prove the second part of this property Applying equation 18 of property 2 for place P_k, at t_0 we obtain $F_{k+1}<F_1.G_1^{k+1}$.
  Since this equation is independent of time its verifies for every time t> t_0.
- Let us prove the third part of the property:
  At t > t_0, from the first part of this property, $B_i(t)$ = 0 for all places P_i between P_k and P_q. According equation (16) to place P_q we have: $j_q(t)=\Phi_{k+1}.G_{k+1}^{q}$ as long as while P_k is marked. Since at t_0 the dynamic balance of P_q, is positive, $\Phi_{q+1}<\Phi_{k+1}.G_{k+1}^{q+1}$ (by property 4).
  But from equation (6) we deduced that at t > t_0:
  $B_q(t)=w_q.\Phi_{k+1}.G_{k+1}^{q}-v_q\Phi_{q+1}$
  so $B_q(t)>w_q.\Phi_{k+1}.G_{k+1}^{q}-v_q\Phi_{k+1}.G_{k+1}^{q+1}=0$
  hence this P_q place remain positive while P_k is marked.

❑

A formal definition of liveness property of a continuous weighted elementary circuit can be now given.

4.2 Liveness of a continuous weighted circuit

In the previous section, dedicated to continuous weighted marked path, we have given limits and exact values of instantaneous firing speeds. In the following, we will prove that liveness property of a continuous weighted circuit depends on the value of the loop gain.

Theorem 1*:* A ***continuous weighted elementary circuit is live*** if and only if it is generating or neutral and has at least one place P_i with not null marking ($m_i^0>0$).

❑

Proof: Let us consider an elementary circuit γ composed of n places and n transitions.

Two cases are distinguished:

Case 1: every place of the circuit is empty, at time t, i.e., $\forall$ $P_i \in \gamma$, m_i (t) = 0.

Hence all transitions are not enabled. Then the circuit is not live.

Case 2: In the following, we consider the initial marking where exactly one place is marked (all the others are supposed empty). In fact, if the elementary circuit is live in this case, it will be live for all markings greater than this one, and therefore for a marking with several marked places.

There exists, in the circuit γ = <P_n, T_1, P_1, T_2, P_2, …, T_n, >, exactly one marked place (all the others are supposed empty), noted P_n, at time t_0, i.e.:

$\exists$ $P_n \in \gamma$, / $m_n(t_0)$ = ε > 0 and $\forall$ $P_{j\neq i} \in \gamma$, $m_j(t_0)$ = 0.

In this case, to prove that an elementary continuous circuit is live, let us prove that every transition is enabled at time t and stays enabled whatever the future evolution.

At time t_0 : we deduce from property 1, applied to the circuit with i = n and q = n:

$j_1(t_0)$ = Φ_1 , and

for j = 2, …, n,

$$j_j(t_0)=\min\left(\Phi_j,\min_{k=i+1}^{j-1}\left(\Phi_k.G_k^j\right)\right) \quad (20)$$

Consequently the instantaneous firing speed vector is strictly positive, $j(t_0)>0$, at the initial date.

For the empty places P_j, ($\forall$ $P_{j\neq i} \in \gamma$, $m_j(t_0)$ = 0), by definition (see remark 2 section 2) they can not have a negative balance. Thus, each dynamic balance of an empty place is positive or null.

Consequently, every transition of the circuit is enabled at time t_0, and dynamic balances of all places, excepted place P_n, are positive or null.

However, at time t_0, the dynamic balance of place P_n depends on the value of the loop gain G(γ).

a) if G(γ) < 1 (the circuit is absorbing),

** At time t_0:*

According to equation (15) applied at transition T_n

$j_n(t_0) \le F_1.G_1^n$. (21)

From equation (6), $B_n(t_0) = -v_n.F_1 + w_n.j_n(t_0)$.

thus, $B_n(t_0) \le F_1.(w_n G_1^n - v_n)$. (22)

As the loop gain is supposed to be strictly inferior to one, we deduce from eq. (5), $G(g) = \frac{w_n}{v_n} G_1^n < 1$,

thus the dynamic balance of place P_n is negative, $B_n(t_0)<0$.

In summary, at time t_0, for an absorbing continuous circuit γ (G(γ) < 1):

φ_j (t_0) > 0 ∀ T_j ∈ γ,

$B_n(t_0)$ < 0 for P_n ∈ γ with m_n (t_0) > 0,

$B_j(t_0)$ > 0 or $B_j(t_0)$ = 0 for P_j ∈ γ , j ≠ n.

In other terms, after t_0, the marking of place P_n decreases, the marking of downstream places stays null or increases.

We call P_q the first place from P_n, which has a positive dynamic balance at time t_0.

From this initial date t_0, the next event will occur when marking of place P_n will be equal to zero. We note this instant t'.

** Between these two dates [t_0, t'[:*

Between two event dates, all instantaneous firing speeds and all dynamic balances are constant. Then $B_n(t)$ is negative for t_0 ≤ t < t'. For places, which have a positive dynamic balance, their marking increases, the others stay empty.

** For time t ³ t':*

At time t': place P_n becomes empty and its dynamic balance will stay null for all t > t'. It sufficient to apply the first part of property 4 to the path from P_k (first upstream marked place to place P_n at t') to P_q (first downstream marked place to place P_n at t'.

We decompose at time t', the circuit into one-marked paths, and deduce from property 1 that every transition is enabled (strongly or weakly).

Now, the first place P_q, having a positive dynamic balance at time t_0 is marked.

$B_q(t') = -v_q.\Phi_{q+1} + w_q.j_q(t)$

According to property 3, for the path r = <P_q, T_{q+1},…, P_{q-1}, T_q>, the instantaneous firing speed of transition T_q is limited by: $j_q(t') \le \Phi_{q+1}.G_{q+1}^q$.

So, $B_q(t') \le \Phi_{q+1}.(w_q G_{q+1}^q - v_q)$.

With the same reasoning used for the place P_n at time t_0, we deduce (as G(γ) < 1) that the dynamic balance of place P_q is negative, B_q(t') < 0 and will stay negative or null whatever the future evolution.

Nevertheless, the positive dynamic balance of other places stays positive.

In summary, at time t' (and until the next event), for an absorbing continuous circuit γ (G(γ)<1): m_n (t') = 0

φ_j (t') > 0 ∀ T_j ∈ γ,

B_i(t') = 0 for i = n, …q-1,

B_q(t') < 0

B_j(t') > 0 or $B_j(t_0)$ = 0 for P_j ∈ γ, j=q+1,…, n-1.

We call $P_{q'}$ the first place from P_q, which has a positive dynamic balance at time t_0.

We denote by t", the instant when the marking of place P_q will be equal to zero.

** At time t":* place P_q becomes empty and its dynamic balance will stay null for all t > t".

With the same way, for an absorbing continuous circuit γ (G(γ)<1), at time t'': m_q(t")= 0

φ_j (t") > 0 ∀ T_j ∈ γ,

B_i(t") = 0 for i = n, …q'-1,

$B_{q'}$(t") < 0

B_j(t") > 0 or $B_j(t_0)$=0 for P_j ∈ γ , j=q'+1,…, n-1.

With the same reasoning for all places, which can be marked, one after one, their marking will be equal to zero, and the circuit will be empty. Consequently, at this time (a maximum of p events where p represents the number of positive balances at the initial date), none transition is enabled and the circuit is deadlocked.

Therefore, if G(γ) < 1 then the circuit is not live.

b) G(γ) ≥ 1 (the circuit is generating or neutral).

** at time t_0:* m_n (t_0) > 0,

φ_j (t_0) > 0 ∀ T_j ∈ γ,

$B_j(t_0)$ > 0 or $B_j(t_0)$ = 0 for P_j ∈ γ , j ≠ n.

To study the dynamic balance of P_n ,we are going to consider two cases:

b1. $B_n(t_0)$ ³ 0:

As all other places have positive or null dynamic balance at t_0, we have $B(t_0)$ ≥ 0. Then, it is the steady state. Therefore, in this case, the circuit is live.

b2. $B_n(t_0)$ < 0:

** at time t_0:* Let us prove, by absurd, that it exists at least one place P_i such that $B_i(t_0)$ > 0.

If $m_i(t_0)$ = 0 and $B_i(t_0)$ = 0 then $j_{i+1}(t_0) = \frac{w_i}{v_i}.j_i(t_0)$.

Therefore, if we suppose that every empty place P_i, i = 1, …, n-1, has a null balance, $B_i(t_0) = 0$, from property 2, we have $\varphi_n(t_0) = \varphi_1(t_0).G_1^n = \Phi_1.G_1^n$.
So, $B_n(t_0) = w_n.\varphi_n(t_0) - v_n.\varphi_1(t_0) = w_n.G_1^n \varphi_1(t_0) - v_n.\varphi_1(t_0)$
Since $G(\gamma) \geq 1$, we deduce from this last equation that $B_n(t_0) \geq 0$ which is in contradiction with the assumption, $B_n(t_0) < 0$. Then it exists at least one place P_i such that $B_i(t_0) > 0$.

We note P_k, the first upstream place to T_n, which has a positive balance at time t_0, and t' the instant when marking of place P_n becomes equal to zero.

In summarise, at time t_0 and until the next event, this equations are verified:

$\varphi_j(t_0) > 0 \ \forall \ T_j \in \gamma$,
$B_n(t_0) < 0$ for $P_n \in \gamma$ and $m_n(t_0) > 0$
$B_j(t_0) > 0$ or $B_j(t_0) = 0$ for $P_j \in \gamma$, j= n+1,…, k-1.
$B_k(t_0) > 0$
$B_i(t_0) = 0$ for $P_i \in \gamma$, i = k+1,…, n-1

* *For time t" > t ≥ t':* (t" corresponds to instant of the next event).

Firstly, let us prove that the dynamic balance $B_k(t)$ marked place P_k, stays non negative for all t ≥ t'. We consider two cases:

- all the other places are empty, at time t'.
  Property 1 applied to T_k: $\varphi_k(t) = G_{k+1}^k.\Phi_{k+1}$, ∀ t≥t'
  and since $G(\gamma) \geq 1$,
  $B_k(t) = w_k.G_{k+1}^k.\Phi_{k+1} - v_{k+1}.\Phi_{k+1} \geq 0$ ∀ t ≥ t'.
  Then, P_k is always marked and this is the steady state. The circuit is then live .

- there is at least one other marked place, at time t'.
  We note P_q, the first upstream marked place to T_k at time t = t'. From property 4 the dynamic balance of P_k stays positive as long as P_q is marked (at least, until t"). Then until the next event t", P_k stays marked and the circuit is live.

By the same analyse for each step (each event), the marking of place P_k is always positive for all time t > t_0, transition T_k is always strongly enabled and the other transitions are also enabled (weakly or strongly). Consequently, a neutral or generating circuit is live.

❑

In the case of neutral circuit ($G(\gamma) = 1$) and under some marking assumptions, a final steady state is reached and expression of the final instantaneous firing speed vector is given.

Corollary: Let a neutral continuous weighted circuit γ with exactly one initial marked place P_n (all the others are supposed empty). If the continuous circuit is live (and neutral), after a finite time t_f, its behaviour reaches a steady state and final instantaneous firing speeds are given by:

for j = k+1, $\varphi_{k+1}(t) = \Phi_{k+1}$, $\forall \ t \geq t_f$
for j ≠ k+1, $\varphi_j(t) = G_{k+1}^j.\Phi_{k+1}$, $\forall \ t \geq t_f$

where P_k is the first upstream place from P_n such that $B_k(t_0) > 0$ (In case that none place verifies this condition, the previous place P_k is then place P_n).

❑

Proof: From property 2, since all places are empty at time t_0, we have: $\varphi_n(t_0) \leq \Phi_1.G_1^n$.
As the loop gain is equal to one ($G(\gamma) = 1$), we have
$B_n(t_0) \leq w_n.\Phi_1 G_1^n - v_1.\Phi_1 = 0$
We are going to consider two cases:

* $B_n(t_0) = 0$:

We will prove that all dynamic balances are null at time t ≥ t_0.

We assume in the neutral circuit, that a place P_q has its dynamic balance positive. At t = $t_0+\varepsilon$, P_q is marked and from property 3, $\varphi_n(t) \leq \Phi_{q+1}.G_{q+1}^n$ and $\varphi_q(t) \leq \Phi_1.G_1^q$. Since $B_n(t) = w_n.\varphi_n(t) - v_n.\Phi_1 = 0$, we obtain $\varphi_q(t) \leq \varphi_n(t).G_n^q$. As $G(\gamma) = 1$, we deduce:

$$B_q(t) = w_q.\varphi_q(t) - v_q.\Phi_{q+1} \leq w_q.\varphi_n(t) G_n^q - v_q.\frac{\varphi_n(t)}{G_{q+1}^n} = 0$$

It is in contradiction with the assumption $B_q(t) > 0$.
Then, all dynamic balances (excepted for place P_n) are null, and the steady state is reached at time $t_f = t_0$.
From property 2, we obtain :

$\varphi_1(t) = \Phi_1$, ∀ t ≥ t_0
for j ≠ 1, $\varphi_j(t) = G_{k+1}^j.\Phi_{k+1}$, ∀ t ≥ t_0

* $B_n(t_0) < 0$:

According to case b2 (see proof of theorem1), there is a least one place with a positive balance. We note P_q, the first downstream place to place P_n with $B_q(t_0) > 0$ and we call P_k, the first upstream place to place P_n with $B_k(t_0) > 0$ (remark : P_k can be place P_q). Let us prove that this dynamic balance $B_q(t)$ will be negative at time t' (t' is the instant where $B_n(t_0) = 0$).

At time t', from property 4, all places between T_{k+1} and T_q are empty and their dynamic balance equals to 0.
From property 3 (eq.19): $\varphi_q(t) \leq \Phi_{k+1}.G_{k+1}^q$
and $\Phi_{k+1} < \Phi_{q+1}.G_{q+1}^{k+1}$

Then, $j_q(t) < \Phi_{q+1}.G_{q+1}^q$ and $j_q(t).\frac{w_q}{v_q} < \Phi_{q+1}.G_q^q$.

As $G_q^q = G(g) = 1$, we obtain: $j_q(t).\frac{w_q}{v_q} < \Phi_{q+1}$.

thus, $B_q(t) = w_q.j_q(t) - v_q.\Phi_{q+1} < 0$ for t∈[t', t"[, such that t" is the instant where $m_q(t) = 0$.

By the same reasoning, the marking of all marked places following P_q and different of P_k, are going to be null. Finally, at a finished time t_f, only place P_k will be marked and its dynamic balance will be null. In fact, from equation (16) of property 2, we have:

$j_k(t_f) = F_{k+1}.G_{k+1}^k$ .then (as G(γ) = 1)

$B_k(t_f) = w_k.j_k(t_f) - v_k.F_{k+1} = w_k.F_{k+1}.G_{k+1}^k - v_k.F_{k+1} = 0$

So, transition T_{k+1} is strongly enabled and the other transitions are weakly enabled.

Therefore, according property 2, we deduce:

for $j = k+1,\ j_{k+1}(t) = F_{k+1},\ " t ³ t_f$

for $j ¹ k+1,\ j_j(t) = G_{k+1}^j.\Phi_{k+1},\ " t ³ t_f$

5 Liveness of continuous weighted graphs

Finally, by decomposing a strongly connected continuous marked graph into elementary continuous circuits, we can generalise the condition of liveness.

Theorem 2: A strongly connected continuous weighted marked graph is live if and only if all its elementary circuits are generating or neutral and have at least one place P_i with not null marking ($m_i^0 > 0$).

❑

Proof: A strongly connected continuous weighted marked graph is live if and only if all its elementary circuits are live. Therefore, from the theorem 1, it is live if and only if all its elementary circuits are generating or neutral and have at least one place P_i with not null marking m_i^0.

6 Conclusion

Since many years, the problem of liveness in Petri nets has been studied as a basic behavioural property. In this paper, we provide novel criteria of liveness of continuous weighted circuits and continuous weighted marked graphs. For neutral circuit and under marking conditions, we characterise the (permanent) steady state of the continuous system by given the final firing speed values.

It would seem, therefore, that further investigations are needed in order to find liveness of hybrid PNs. In fact, hybrid PNs [6], composed by T-timed discrete PNs and continuous PNs, are suitable models to analyse hybrid systems as high throughput or batch manufacturing systems. As the matter of fact, the current and continuing focus on a discrete events- continuous time approach in Petri nets will pave the way towards the performance evaluation and performance control of hybrid systems.

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