Estimation of the Basin of Attractions of Stable Equilibrium Points in Full Range Cellular Neural Networks

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Abstract: An approach is discussed to estimate the basin of attraction of stable equilibrium points in linear systems, operating in a saturated mode (LSSM). The approach is valid for Full Range Cellular Neural Networks (FR CNNs) too as far as the dynamic properties of both systems are qualitatively similar. The approach is an extension to previous work on original CNNs. The used technique is based on the determination of the so called tree of regions for each stable equilibrium point and the additional separation of the regions where the boundaries between different basins are located. The obtained trees for each stable equilibrium point give the corresponding basin of attraction.

Key-Words: Cellular Neural Networks (CNN's), Full Range CNN, Linear System Operating in a Saturated Mode (LSSM), basin of attractions *IMACS/IEEE CSCC'99* Proceedings, Pages:2771-2774

1 Introduction

Cellular Neural Networks (CNNs) have been introduced by Chua and Yang as a local interconnected network of non-linear dynamic cells [1-2]. The behaviour of the CNN is defined by the template matrices *A* and *B* and the template vector *I*. In principle CNNs are non-linear dynamic systems with piece-wise linear non-linearity's, which give rise to complex performances. The dynamic properties of this kind of systems are considered in [3] (and in the references given there), [4], [5], [6], [11], [12]. A complete description on the basin of attractions for a simple 2 cells CNN is given in [6]. In [7] and [8] the authors suggest an approach, based on the separation of the state space, to estimate the basin of attractions for the original Chua-Yang CNN.

In 1992, Rodriguez-Vázquez introduced [9] a new model to overcome several drawbacks related to the electronic implementation of the original CNN. One of these drawbacks is the unbounded state variable. This may complicate VLSI implementation, whereas the state variable in Rodriguez-Vázquez' Full Range (FR) model is always bounded between -1 and +1 (normalised), independent of the templates. In [10] Perfetty proposes a circuit which does not reproduce exactly the state equations of the original Chua-Yang CNN. This circuit can be described as a liner system operating in a saturated mode (LSSM). However the FR CNN and the circuit from [10] can be used efficiently to implement the Chua-Yang CNN dynamics, since they share the most significant qualitative properties. In what follows we will consider the FR CNN as a LSSM as far as the dynamic properties of both systems are qualitatively similar. Here, the approach in [8] and [9] will be extended to cover the estimation of the basin of attractions in LSSM and therefore in the FR CNN too.

The paper is outlined as follows. In section 2, some basic results that we will use later are given. In section 3 we show the applicability of the approach in [7] and [8] to estimate the basin of attractions in LSSM systems. In section 4 we give an example to illustrate the outlined approach and some conclusions are given in section 5.

2 Preliminaries

The original Chua-Yang continuous-time CNN [1] is described by the non-linear ordinary differential equation in vector-matrix form:

$$\begin{aligned}\mathbf{dx(t)/dt} &= -\mathbf{x(t)} + \mathbf{Ay(t)} + \mathbf{b} = \\ &-\mathbf{x(t)} + \mathbf{Af(x(t))} + \mathbf{b}\end{aligned} \qquad (1)$$

Here $\mathbf{x(t)}=(x_1(t), x_2(t), ..., x_n(t))^T$ is the state vector at time t, **A** is the feedback matrix, **b** represent the time-invariant inputs and the threshold [1], [5] and $\mathbf{y=f(x)}=(f(x_1), f(x_2), ..., f(x_n))^T$, where f(.) is the piece-wise linear function

$$y_i=f(x_i)=0.5(|x_i+1|+|x_i-1|) \quad (2)$$

The Rodriguez-Vázquez Full Range (FR) model [9] is described by the system of equations in the following vector-matrix form:

$$\mathbf{dx(t)/dt=-g[x(t)]+Ay(t)+b} \quad (3)$$

where $\mathbf{g(x)}=(g(x_1), g(x_2), ...,g(x_n))^T$ and

$$g(x_i)=\begin{cases} -m(x_i+1)+1, & \text{if } x_i<1 \\ -x_i & \text{otherwise} \\ -m(x_i-1)-1 & \text{if } x_i>1 \end{cases} \quad (4)$$

Here, the dynamic route of a single cell is like the Chua-Yang model [1] when m=1. As can be seen [9], the state variables cannot be larger than +1 or -1 if m= ∞. If any state enters the boundaries of the hypercube $x_i \in [-1, 1]$, i=1,2,…,n, the state will be "pushed back" infinitely fast due to infinite m. As it was stated in [9] the Full Range model provides correct results, qualitatively similar to those of the Chua-Yang model (for all templates considered there) and can be used efficiently to implement the Chua-Yang CNN dynamics. Furthermore, the dynamics of the FR CNN are similar to the Perffety circuit [10], which can be described by a linear system operating in a saturated mode (LSSM) [11]

$$\mathbf{dy(t)/dt=A_L y(t)+b,\ y} \in D^n \quad (5)$$

where $D^n=[-1,1]^n$ is a hypercube i.e.

$$y_i \in [-1, 1],\ i=1,2,\dots,n.$$

In the case of equivalent FR CNN $\mathbf{A_L=A-I}$ and equation (5) has the form

$$\mathbf{dy(t)/dt =(A-I)y(t)+b,\ y} \in D^n \quad (6)$$

Like in [7], [8] and [12] the state space of system (6) could be partitioned into 3^n disjoint regions which are classified as follows:

a) linear region DO where $y_i \in (-1, 1)$, for every i;
b) 2^n saturation regions DS, where y_i is +1 or -1 for every i;
c) 3^n-2^n-1 partial saturation regions DP, where for some i, y_i is +1 or -1 and $y_i \in (-1, 1)$ for the others.

In contrast to [7], [8] and [12] the saturation and partial saturation regions are degenerate e.g. partial saturation regions are finite (do not extend to infinity) while the saturation regions are the verteces of the hypercube D^n. Some basic properties and theorems [4], [10], [11] for the LSSM (5) are outlined below.

Property 1. If A_L is not negative definite, the eventual equilibrium points in the linear region is unstable.

Property 2. If the reduced matrices corresponding to the partial saturation regions are not negative definite, the eventual equilibrium points in the partial saturation regions are unstable

Property 3. In the saturation regions: if $\min((A_L\xi+b)*\xi)<0$, then $y=\xi$ is not an equilibrium point of (5). If $\min((A_L\xi+b)*\xi)>0$, then ξ is an asymptotically stable equilibrium point of (5), where the operation "*" is defined as $\mathbf{a*b}=(a_1b_1,...,a_nb_n)^T$.

Property 4. If A_L is symmetric, each solution of (5) converges to an equilibrium point of (5).

Applying Properties 1-4 to system (6) describing the behaviour of the LSSM equivalent to FR CNN the following theorems can be proved.

Theorem 1. If $a_{ii}>1$, every equilibrium point inside the linear region and the partial saturation regions is unstable.

Proof: If $a_{ii}>1$, all the diagonal elements of A-I are positive. Since the trace is equal to the sum of the real part of the eigenvalues, at least one eigenvalue of A-I has positive real part. Thus the eventual equilibrium point inside the linear region is unstable (Property 1). The same holds for the reduced matrices corresponding to the partial saturation regions.

Theorem 2. If the LSSM is convergent and $a_{ii}>1$, for almost every initial condition the steady state is binary: $y_i(\infty) \in \{-1, 1\}$ for every b.

Proof: If the LSSM is convergent, the trajectory converges to a fixed point as $t\rightarrow\infty$ for every initial state [5]. Owing to presence of noise, unstable equilibra are not observable in practice and from Theorem 1 follows that the steady state is one of the saturation regions i.e. it is binary.

Theorem 3. Let ξ belongs to the saturation regions DS. An asymptotically stable equilibrium point $y=\xi$ exists, iff

$$\sum_{j=1}^{n} a_{ij}\xi_j\xi_i + b_i\xi_i > 1\ ,\ i=1,...,n \quad (7)$$

Proof: The result follows after applying Property 3 to system (6), as regards to stable equilibra.

Theorem 4. If $a_{ii}>1$, there is one to one-correspondence between the asymptotically stable equilibra of the LSSM model (6) and those of the CNN of equations (1) and (2).

Proof: As proved in Theorem 1, the equilibra inside the linear and partial saturation regions are unstable. In practice, only the equilibra that are asymptotically stable are of interest, i.e. the equilibra corresponding to any saturation region. They coincide with the solutions of the linear constrains (7). These equilibra are the same as those of the CNN of equations (1) and (2) in the sense that the corresponding output variables are coinsident.

Theorem 5. A LSSM model of the CNN with a symmetric template is convergent, i.e. $y_i(t) \to y_i(\infty) \in \{-1, 1\}$ for i=1,...,n and for every initial condition.

Proof: See Property 4.

3 Description of the approach

The application of the approach from [7] and [8] for the case of LSSM (6) is based on Theorems 1-5 given in the previous section. First, following Theorem 3 one should find the stable equilibrium points of the system. Later the technique as described in [7] will be applied in order to construct trees of regions connected to a stable equilibrium point in each saturation region. Then for the regions in which boundaries between different basin of attractions are located one should find the description of the internal hyperplanes [8]. These hyperplanes (internal boundaries) could not be crossed by state trajectories and then a new subset of regions can be derived using the separation technique [8]. After a new set of regions is obtained, one simply should apply again the approach in [7] in order to construct the final trees.

Here the saturation regions coincide with vertexes of the hypercube and the partial saturation regions are in fact hyperplanes and hyperlines enclosing the hypercube. As was mentioned the partial saturation regions have a reduced order in comparison with the partial saturation regions in the Chua-Yang model. Therefore both the corresponding calculations for evaluation the system behaviour on the boundaries between different regions and calculations for obtaining the separation hyperplanes in the regions where the boundaries between different basins are located are easier than in [7] and [8]. Once constructed (following the rules in [7] and [8]) the tree represent exactly the basin of attraction of each stable equilibrium point. Here it is possible to fulfil precisely the stopping criteria [7] when the trees are constructed.

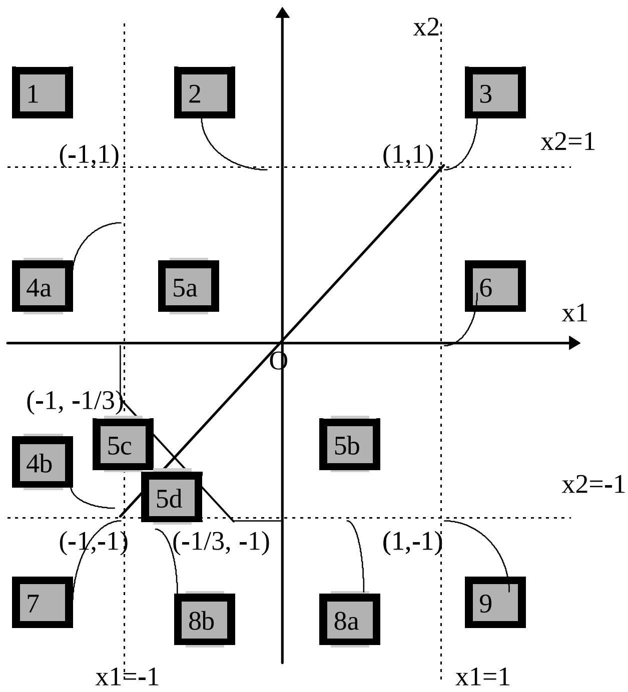

Figure 1: Regions for separation the state space for given example

4 Example

Let us consider a two cell CNN described by the system of equations [7], [8]

$$\begin{aligned} dx_1/dt &= -x_1 + 1.5y_1 + y_2 + 1 \\ dx_2/dt &= -x_2 + y_1 + 1.5y_2 + 1 \end{aligned} \tag{8}$$

and the corresponding LSSM

$$\begin{aligned} dy_1/dt &= 0.5y_1 + y_2 + 1 \\ dy_2/dt &= y_1 + 0.5y_2 + 1 \end{aligned} \tag{9}$$

where $y_i \in [-1, 1]$, i=1,2.

The above CNN has two stable equilibrium points, namely E1=(x_1,x_2)=(3.5, 3.5) and E2=(x_1,x_2)=(-1.5, -1.5). The corresponding stable equilibrium points for (8) are E1=(y_1,y_2)=(1, 1) and E2=(y_1,y_2)=(-1, -1). The separated state space is given in Figure 1. The state space and the partitioning are similar to those in [8] but now the partial saturation regions are four separation lines and the four saturation regions are the points (-1,-1), (-1,1), (1,-1) and (1,1). Linear region 5 is the same as in [8].

Now, using the obtained set of regions and the rules described in [7] and [8] we can construct the trees for the stable equilibrium points E1 and E2. It is easy to check that the tree for E1 include regions {3, 2, 6, 1, 9, 4a, 8a, 5a and 5b) and the tree for E2 include regions {7, 4b, 8b, 5c and 5d}. In comparison with [8] regions 8a, 8b and 4a, 4b are slightly different than the corresponding ones. For example, region 8b is now the line between (-1, -1) and (-1/3, -1) while region 8a is the line between (-1/3, -1) and (1, -1). This result gives the possibility to describe *exactly* the basin of attractions for each stable equilibrium points E1 and E2.

5 Conclusions

In this paper we suggest a method for determining an estimation of the basin of attractions for the stable equilibrium points in linear system operating in a saturated mode (LSSM). The approach is valid for a Full Range Cellular Neural Networks (FR CNNs) too as far as the dynamic properties of both systems are qualitatively similar. The approach is an extension of the previous authors work for an original CNN and is based on determining the so called tree of regions connected with each stable equilibrium point described in the previous authors work.

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