Control System Synthesis as a Multiple Criteria Decision Making Problem

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Abstract: The process of control system synthesis requires evaluation of a number of alternative control structures with respect to the resulting performance of the closed-loop system. At present, there is not a general method for selection of best alternative for control system.

In this paper, a general problem of control system synthesis is formulated and represented as a multiple criteria decision making problem. Then, an approach is proposed to solve this problem through comparison between alternative control structures and selection of the best one. The developed approach includes three steps. First, the set of all possible alternatives for control structures is defined. Second, for each alternative control system the optimal values of controller parameters are determined. This optimization problem can be solved by applying standard optimization techniques. Third, the best alternative control structure is selected from the set of available alternatives. In order to do it, three types of fuzzy preference relations are introduced that show how "good" the alternative $\boldsymbol{A_i}$ is compared to the alternative $\boldsymbol{A_j}$ with respect to optimization of the multiple criteria, w.r.t. satisfaction of the inequality constraints and w.r.t. satisfaction of the equality constraints. Also, the total fuzzy preference relation is defined. Then, the fuzzy subset of nondominated alternatives is determined and the best alternative is selected as the one whose value of the membership function is maximal. Finally, an approach is described for selection of the weight coefficients needed for the solution of controller synthesis problem.

Key Words: Decision making, Control system synthesis, Multiple criteria, Constraints.
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1. Introduction.

The process of control system synthesis requires evaluation of a number of alternative control structures with respect to the resulting performance of the closed-loop system. At present, there are only few methods which deal with the problem of controller structure selection. Thus, in [1] a frequency domain method has been proposed for controller design for linear plants. According to this method, the optimal control configuration is the simplest one (with the lowest possible order) which gives a satisfactory approximation of the desired closed-loop system. It should be noted, however, that this method can be applied for linear plants only and it is not shown how the tradeoff between controller complexity and controller performance is made. Also, sometimes it is more appropriate to choose control structure that optimizes closed-loop performance (for example, one that minimizes the integral squared error) rather than to select controller which approximates a preliminary specified closed-loop response. At present, there is not a general method for selection of best alternative for control system.

In this paper, a general problem of control system synthesis is formulated and represented as a multiple criteria decision making problem. Then, an approach is proposed to solve this problem through comparison between alternative control structures and selection of the best one.

2. Problem formulation.

The control system under consideration is shown in Fig.1, where $\underline{\boldsymbol{y}}$ is an *ny*-dimensional vector of the variables to be controlled, $\underline{\boldsymbol{y}}_{sp}$ is the set point vector, $\underline{\boldsymbol{u}}$ is an *nu*-dimensional vector of control variables and $\underline{\boldsymbol{n}}$ is an *nn* - dimensional vector of disturbances. It is supposed that plant dynamics is described by the following general equations:

$$\frac{d\underline{\boldsymbol{x}}}{dt} = \underline{\boldsymbol{f}}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{u}}) \quad (1)$$

$$\underline{\boldsymbol{y}} = \underline{\boldsymbol{g}}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{u}}), \quad (2)$$

where $\underline{x}$ is an nx-dimensional vector of state variables, $\underline{f}$ and $\underline{g}$ are vector functions.

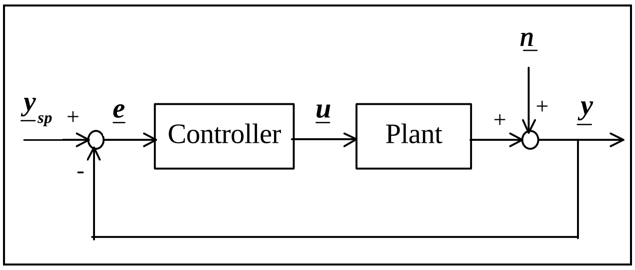

Fig.1. Control system.

The problem to be solved is to find control law:

$$\underline{u} = \underline{u}(\underline{y}) \tag{3}$$

that will maximize the multiple optimality criteria represented in the following general form:

$$I_k(\underline{u}) = G_k[\underline{x}(t_f)] + \int_0^{t_f} f_{0k}(\underline{x}, \underline{y}, \underline{u}) \cdot dt \quad \rightarrow \ \max \qquad k = 1,2,\dots,m \tag{4}$$

subject to constraints of inequality type:

$$y_k(\underline{x}(t), \underline{y}(t), \underline{u}(t)) \le 0 \ , \ t \in [0; t_f] \qquad k = 1,2,\dots,l_1 \tag{5}$$

and constraints of equality type:

$$j_k(\underline{x}(t), \underline{y}(t), \underline{u}(t)) = 0 \ , \ t \in [0; t_f] \qquad k = 1,2,\dots,l_2 \tag{6}$$

Constraints (5) and (6) can include both path constraints that have to be satisfied in the time interval $t \in [0; t_f]$ and final time constraints that have to be kept at the end time t_f. In the further considerations it is more convenient to have only final time constraints in the control problem formulation. For this purpose, the approach proposed in [2] can be applied to represent path constraints as end time constraints. According to [2], path constraints of the form:

$$a_{k\min} \le a_k(\underline{x}(t), \underline{u}(t)) \le a_{k\max} \qquad k = 1,2,\dots,l \tag{7}$$

can be converted to final time constraints through definition of a new state variable as:

$$\frac{dx_{nx+1}}{dt} = (\underline{a}_{\min} - \underline{a})^T \cdot W_1 \cdot (\underline{a}_{\min} - \underline{a}) + (\underline{a} - \underline{a}_{\max})^T \cdot W_2 \cdot (\underline{a} - \underline{a}_{\max}) \tag{8}$$

$$x_{nx+1}(0) = 0 \ ,$$

where $\underline{a} = [a_1(\underline{x}(t), \underline{u}(t)) \ \dots \ a_l(\underline{x}(t), \underline{u}(t))]^T$, $\underline{a}_{\min} = [a_{1\min} \ \dots \ a_{l\min}]^T$, $\underline{a}_{\max} = [a_{1\max} \ \dots \ a_{l\max}]^T$. In (8), W_1 and W_2 are $l \times l$ diagonal weighting matrices whose elements reflect the satisfaction of the path constraints. Then, a new final time constraint is added:

$$x_{nx+1}(t_f) = 0 \tag{9}$$

It is clear that when constraint (9) is satisfied, then constraints (7) are satisfied too.

In the control problem formulated above, path constraints of types (5) and (6) are now represented as final time constraints through definition of the new state variable:

$$\frac{dx_{nx+1}}{dt} = \underline{y}^T(\underline{x}, \underline{y}, \underline{u}) \cdot W_1 \cdot \underline{y}(\underline{x}, \underline{y}, \underline{u}) + \underline{j}^T(\underline{x}, \underline{y}, \underline{u}) \cdot W_2 \cdot \underline{j}(\underline{x}, \underline{y}, \underline{u}) \tag{10}$$

$$x_{nx+1}(0) = 0 \ ,$$

where:

$$\underline{y}(\underline{x}, \underline{y}, \underline{u}) = [y_1(\underline{x}, \underline{y}, \underline{u}) \ \dots \ y_{l_1}(\underline{x}, \underline{y}, \underline{u})]^T \tag{11}$$

$$\underline{j}(\underline{x}, \underline{y}, \underline{u}) = [j_1(\underline{x}, \underline{y}, \underline{u}) \ \dots \ j_{l_2}(\underline{x}, \underline{y}, \underline{u})]^T \tag{12}$$

and the elements of the weighting matrices W_1 and W_2 are:

$$w_{1kk} = 0, \ \mathit{if} \ y_k(\underline{x}, \underline{y}, \underline{u}) \le 0 \ ; \ k = 1,2,\dots,l_1 \tag{13}$$

$$w_{1kk} = 1, \ \mathit{if} \ y_k(\underline{x}, \underline{y}, \underline{u}) > 0 \ ; \ k = 1,2,\dots,l_1 \tag{14}$$

$$w_{2kk} = 0, \ \mathit{if} \ -d_k \le j_k(\underline{x}, \underline{y}, \underline{u}) \le d_k \qquad k = 1,2,\dots,l_2 \tag{15}$$

$$w_{2kk} = 1, \ \mathit{if} \ \ j_k(\underline{x}(t), \underline{y}(t), \underline{u}(t)) < -d_k \ \vee \ j_k(\underline{x}(t), \underline{y}(t), \underline{u}(t)) > d_k \tag{16}$$

$$k = 1,2,\dots,l_2$$

Here, the equality constraints (6) are relaxed by introducing the small positive value d_k. Now, new final time constraint is added:

$$x_{nx+1}(t_f) = 0 \tag{17}$$

When constraint (17) is satisfied, the state and control trajectories are feasible with respect to the inequality and the equality path constraints (5) and (6). Thus, the problem of controller synthesis is to find control law $\underline{u} = \underline{u}(\underline{y})$ that will maximize the multiple optimality criteria (4) subject to the following final time constraints:

- constraints of inequality type:

$$y_k^*(\underline{x}^*(t_f), \underline{y}(t_f), \underline{u}(t_f)) \le 0 \ , \ k = 1,2,\dots,l_1^* \tag{18}$$

- constraints of equality type:

$$j_k^*(\underline{x}^*(t_f), \underline{y}(t_f), \underline{u}(t_f)) = 0 \ , \ k = 1,2,\dots,l_2^* , \tag{19}$$

where $\underline{x}^* = [x_1 \ x_2 \ \dots \ x_{nx} \ x_{nx+1}]^T$ is the augmented state vector.

The problem of control system synthesis consists of choosing an optimal structure and optimal parameters of the controller, i.e. it consists of structural and parametric synthesis. Here, the set of the possible alternatives for controller structures is denoted by:

$$\boldsymbol{A} = \{\boldsymbol{A}_1, \boldsymbol{A}_2, \ \dots \ \boldsymbol{A}_n\} \tag{20}$$

Then, controller parameters for alternative $\boldsymbol{A}_i$ are denoted by vector $\underline{\boldsymbol{p}}(\boldsymbol{A}_i)$ and their optimal values are denoted by vector $\underline{\boldsymbol{p}}_{opt}(\boldsymbol{A}_i)$. Thus, control law can be expressed as:

$$\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}(\underline{\boldsymbol{y}}, \boldsymbol{A}_i, \underline{\boldsymbol{p}}(\boldsymbol{A}_i)) \tag{21}$$

In this way, the problem of controller synthesis for a given plant is to find the best alternative $\boldsymbol{A}_{best}$ for controller structure and the optimal controller parameters $\underline{\boldsymbol{p}}_{opt}(\boldsymbol{A}_{best})$ for this structure so that the criteria (4) are optimized and constraints (18) and (19) are satisfied.

3. Approach for control system synthesis.

Here, an approach is presented to solve the controller synthesis problem formulated above. It includes the following steps:

1. Define the set of all possible alternatives of controller structure:

$$\boldsymbol{A} = \{\boldsymbol{A}_1, \boldsymbol{A}_2, \ \dots \ \boldsymbol{A}_n\} \tag{22}$$

An example set of possible alternatives is shown in Fig.2.

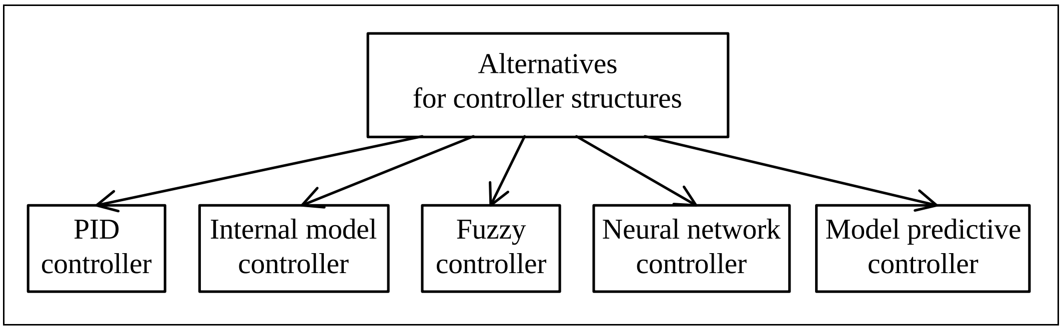

Fig.2. Alternatives for controller structures.

2. Determine the optimal controller parameters $\underline{\boldsymbol{p}}_{opt}(\boldsymbol{A}_i)$ for alternative $\boldsymbol{A}_i$.

These are parameters which maximize the multiple optimality criteria (4) and satisfy constraints (18) and (19) given that controller structure is $\boldsymbol{A}_i$. Here, this multiple criteria optimization problem is solved by applying fuzzy sets theory. Three types of membership functions are introduced:

- membership function $m_{\boldsymbol{I}}(\underline{\boldsymbol{p}})$ that shows how "good" are the values $\underline{\boldsymbol{p}}(\boldsymbol{A}_i)$ of controller parameters with respect to optimization of the multiple criteria $\boldsymbol{I}_k(\underline{\boldsymbol{u}})$, $\boldsymbol{k} = 1,2,\dots,\boldsymbol{m}$;
- membership function $m_{\boldsymbol{y}^*}(\underline{\boldsymbol{p}})$ that shows how "good" are the parameters $\underline{\boldsymbol{p}}$ with regard to satisfying inequality constraints (18);
- membership function $m_{\boldsymbol{j}^*}(\underline{\boldsymbol{p}})$ that shows how "good" are the parameters $\underline{\boldsymbol{p}}$ with respect to satisfying equality constraints (19).

The following steps should be carried out to find the optimal values $\underline{\boldsymbol{p}}_{opt}$ of controller parameters:

2.1. For the current values $\underline{\boldsymbol{p}}$ calculate the membership function $m_{\boldsymbol{I}}(\underline{\boldsymbol{p}})$.

In order to do this, *m* individual membership functions $m_{I_k}(\underline{\boldsymbol{p}})$ correcponding to the *m* criteria to be maximized, are considered:

$$m_{I_k}(\underline{\boldsymbol{p}}) = 1 - \frac{\boldsymbol{I}_{\boldsymbol{k}\max} - \boldsymbol{I}_{\boldsymbol{k}}(\underline{\boldsymbol{p}})}{\boldsymbol{I}_{\boldsymbol{k}\max} - \boldsymbol{I}_{\boldsymbol{k}\min}} \ , \tag{23}$$

where:

$$\boldsymbol{I}_{\boldsymbol{k}\min} = \min_{\underline{\boldsymbol{p}} \in \boldsymbol{P}_0} \boldsymbol{I}_{\boldsymbol{k}}(\underline{\boldsymbol{p}}) \ , \quad \boldsymbol{I}_{\boldsymbol{k}\max} = \max_{\underline{\boldsymbol{p}} \in \boldsymbol{P}_0} \boldsymbol{I}_{\boldsymbol{k}}(\underline{\boldsymbol{p}}) \tag{24}$$

Function (23) shows to which extent the value of the criteria is close to its "best" (maximal) value and it belongs to the interval [0;1]. In (24) $\boldsymbol{P}_0$ denotes the admissible range for searching the optimal values of $\underline{\boldsymbol{p}}$.

Having determined the *m* individual membership functions $m_{I_k}(\underline{\boldsymbol{p}})$, then the function $m_{\boldsymbol{I}}(\underline{\boldsymbol{p}})$ is computed as follows:

$$m_{\boldsymbol{I}}(\underline{\boldsymbol{p}}) = \sum_{\boldsymbol{k}=1}^{\boldsymbol{m}} l_{I_k} \cdot m_{I_k}(\underline{\boldsymbol{p}}) \ , \tag{25}$$

where l_{I_k} is the weight of the *k*-th criterion. These weight coefficients are chosen in a such a way so they will satisfy:

$$\sum_{\boldsymbol{k}=1}^{\boldsymbol{m}} l_{I_k} = 1 \ , \quad l_{I_k} > 0 \tag{26}$$

In section 3, an approach for selection of the weight coefficients is described.

2.2. For the current values $\underline{\boldsymbol{p}}$ calculate the membership function $m_{y^*}(\underline{\boldsymbol{p}})$.

Similarly to the previous step, $\boldsymbol{l}_1^*$ individual membership functions $m_{y^*_k}(\underline{\boldsymbol{p}})$ correcponding to the $\boldsymbol{l}_1^*$ constraints of type (18), are considered:

$$m_{y^*_k}(\underline{\boldsymbol{p}}) = \begin{cases} 0, \ \boldsymbol{if} \ \ y^*_k(\underline{\boldsymbol{p}}) > 0 \\ \\ \dfrac{d_k - y^*_k(\underline{\boldsymbol{p}})}{d_k - y^*_{k\min}}, \ \boldsymbol{if} \ \ y^*_k(\underline{\boldsymbol{p}}) \le 0 \end{cases} \tag{27}$$

where:

$$y^*_{k\min} = \min_{\underline{\boldsymbol{p}} \in \boldsymbol{P}_0} y^*_k(\underline{\boldsymbol{p}}) \tag{28}$$

and $\boldsymbol{d_k}$ is a small positive value that prevents the denominator in (27) from being equal to zero. Function (27) shows how far is the value of $y^*_k(\underline{\boldsymbol{p}})$ from its maximal allowed value (zero) and it belongs to the interval [0;1]. Function $m_{y^*}(\underline{\boldsymbol{p}})$ is now computed:

$$m_{y^*}(\underline{\boldsymbol{p}}) = \sum_{\boldsymbol{k}=1}^{l_1^*} l_{y^*_k} \cdot m_{y^*_k}(\underline{\boldsymbol{p}}) \tag{29}$$

where $l_{y^*_k}$ is the weight of the *k*-th constraint.

2.3. For the current values $\underline{\boldsymbol{p}}$ calculate the membership function $m_{j^*}(\underline{\boldsymbol{p}})$.

Analogously, $\boldsymbol{l}_2^*$ individual membership functions $m_{j^*_k}(\underline{\boldsymbol{p}})$ correcponding to the $\boldsymbol{l}_2^*$ constraints of type (19), are considered:

$$m_{j^*_k}(\underline{\boldsymbol{p}}) = \begin{cases} 0, \ \boldsymbol{if} \ \ j^*_k(\underline{\boldsymbol{p}}) < -d_k \vee j^*_k(\underline{\boldsymbol{p}}) > d_k \\ \\ 1 - \dfrac{\left| j^*_k(\underline{\boldsymbol{p}}) \right|}{d_k}, \ \boldsymbol{if} \ \ -d_k \le j^*_k(\underline{\boldsymbol{p}}) \le d_k \end{cases} \tag{30}$$

where $\boldsymbol{d_k}$ is a small positive value which is used to relax the equality constraints (19). Function (30) shows how "good" is the value of $j^*_k(\underline{\boldsymbol{p}})$ compared to its "best" (zero) value and it belongs to the interval [0;1]. Function $m_{j^*}(\underline{\boldsymbol{p}})$ is computed as follows:

$$m_{j^*}(\underline{\boldsymbol{p}}) = \sum_{\boldsymbol{k}=1}^{l_2^*} l_{j^*_k} \cdot m_{j^*_k}(\underline{\boldsymbol{p}}) \tag{31}$$

where $l_{j^*_k}$ is the weight of the *k*-th constraint.

2.4. For the current values $\underline{\boldsymbol{p}}$ calculate the membership function $m_S(\underline{\boldsymbol{p}})$.

This function shows to what extent $\underline{\boldsymbol{p}}$ can be accepted as the optimal solution of the problem, i.e. how "good" $\underline{\boldsymbol{p}}$ is with respect to optimization of the multiple criteria and satisfaction of all constraints of type (18) and (19). It is computed as follows:

$$m_S(\underline{\boldsymbol{p}}) = l_I \cdot m_I(\underline{\boldsymbol{p}}) + l_{y^*} \cdot m_{y^*} + l_{j^*} \cdot m_{j^*} \ , \tag{32}$$

where l_I, l_{y^*} and l_{j^*} are weight coefficients reflecting the importance respectively of optimality criteria, inequality constraints and equality constraints. They should satisfy:

$$l_I + l_{y^*} + l_{j^*} = 1 \tag{33}$$

2.5. Determine the optimal values $\underline{\boldsymbol{p}}_{opt}$ by maximizing $m_S(\underline{\boldsymbol{p}})$.

The optimal values $\underline{\boldsymbol{p}}_{opt}$ are found through solution of the following optimization problem:

$$\underline{\boldsymbol{p}}_{opt} = \arg \max_{\underline{\boldsymbol{p}} \in \boldsymbol{P}_0} m_S(\underline{\boldsymbol{p}}) \tag{34}$$

Problem (34) can be solved by applying standard optimization techniques. In order to execute this step, all previous steps (from 2.1 to 2.4) have to be repeated several times.

3. Determine the best alternative A_{best} from the set of alternatives $A = \{A_1, A_2, \ \dots \ A_n\}$.

The problem of best alternative selection represents a multiple criteria decision making problem under constraints. Here, a strategy is proposed to solve this problem by introducing three types of fuzzy preference relations:

- preference relation $\boldsymbol{Q_I}$ with respect to the *m* criteria to be optimized:

$$\boldsymbol{Q_I} = \{(\boldsymbol{A}_i, \boldsymbol{A}_j) : \boldsymbol{A}_i, \boldsymbol{A}_j \in \boldsymbol{A}, \ m_I(\boldsymbol{A}_i, \boldsymbol{A}_j)\} \tag{35}$$

- preference relation $\boldsymbol{Q}_y^*$ with respect to the $\boldsymbol{l}_1^*$ inequality constraints to be satisfied:

$$\boldsymbol{Q}_{y^*} = \{(\boldsymbol{A}_i, \boldsymbol{A}_j) : \boldsymbol{A}_i, \boldsymbol{A}_j \in \boldsymbol{A}, \ m_{y^*}(\boldsymbol{A}_i, \boldsymbol{A}_j)\} \tag{36}$$

- preference relation $\boldsymbol{Q}_j^*$ with respect to the $\boldsymbol{l}_2^*$ equality constraints to be satisfied:

$$\boldsymbol{Q}_{j^*} = \{(\boldsymbol{A}_i, \boldsymbol{A}_j) : \boldsymbol{A}_i, \boldsymbol{A}_j \in \boldsymbol{A}, \ m_{j^*}(\boldsymbol{A}_i, \boldsymbol{A}_j)\} \ , \tag{37}$$

where m_I, m_{y^*} and m_{j^*} are membership functions. Function $m_I(\boldsymbol{A}_i, \boldsymbol{A}_j)$ shows how "good" is alternative A_i compared to alternative A_j with respect to optimization of the multiple criteria (4), function $m_{y^*}(A_i, A_j)$ reflects how "good" is A_i in comparison to A_j with regard to satisfaction of the inequality constraints (18) and $m_{j^*}(A_i, A_j)$ shows how "good" is A_i compared to A_j with respect to satisfaction of the equality constraints (19).

The following steps are to be carried out in order to select the best alternative A_{best} of controller structure:

3.1. Determine the fuzzy preference relation with respect to the criteria to be optimized.

In order to determine this preference relation, *m* individual preference relations Q_{I_k} corresponding to the *m* criteria (4) to be optimized, are considered:

$$Q_{I_k} = \{(A_i, A_j) : A_i, A_j \in A,\ m_{I_k}(A_i, A_j)\} \quad k = 1,2,\dots,m \qquad (38)$$

where it is proposed for the membership function $m_{I_k}(A_i, A_j)$ to be computed as follows:

$$m_{I_k}(A_i, A_j) = \begin{cases} 0, \ \textit{if}\ I_k(A_i) < \bar{I}_k \wedge I_k(A_j) > \bar{I}_k \\ 1, \ \textit{if}\ I_k(A_i) > \bar{I}_k \wedge I_k(A_j) < \bar{I}_k \\ \dfrac{\left(I_k(A_i) - \bar{I}_k\right)}{\left(I_k(A_i) - \bar{I}_k\right) + \left(I_k(A_j) - \bar{I}_k\right)}, \\ \quad \textit{if}\ I_k(A_i) > \bar{I}_k \wedge I_k(A_j) > \bar{I}_k \\ 1 - \dfrac{\left(\bar{I}_k - I_k(A_i)\right)}{\left(\bar{I}_k - I_k(A_i)\right) + \left(\bar{I}_k - I_k(A_j)\right)}, \\ \quad \textit{if}\ I_k(A_i) < \bar{I}_k \wedge I_k(A_j) < \bar{I}_k \end{cases} \qquad (39)$$

Here, $\bar{I}_k$ is the average value of the criterion I_k among all available alternatives, i.e.:

$$\bar{I}_k = \frac{1}{n} \cdot \sum_{i=1}^{n} I_k(A_i) \qquad (40)$$

The membership function $m_{I_k}(A_i, A_j)$ shows how "good" the alternative A_i is compared to the alternative A_j with respect to the maximization of criterion I_k. The average value $\bar{I}_k$ of the criterion serves as a basis for comparison.

Having determined the *m* individual preference relations Q_{I_k}, then the membership function of the total preference relation Q_I with respect to all optimization criteria is computed as follows:

$$m_I(A_i, A_j) = \sum_{k=1}^{m} l_{I_k} \cdot m_{I_k}(A_i, A_j), \qquad (41)$$

where l_{I_k} is the weight of the *k*-th criterion. These weight coefficients are chosen in a such a way so they will satisfy:

$$\sum_{k=1}^{m} l_{I_k} = 1, \quad l_{I_k} > 0 \qquad (42)$$

3.2. Determine the fuzzy preference relation with respect to the inequality constraints to be satisfied.

In order to determine this preference relation, l_1^* individual preference relations $Q_{y_k^*}$ corresponding to the l_1^* inequality constraints (18) to be satisfied, are considered:

$$Q_{y_k^*} = \{(A_i, A_j) : A_i, A_j \in A,\ m_{y_k^*}(A_i, A_j)\} \quad k = 1,2,\dots,l_1^* \qquad (43)$$

where it is proposed for the membership function $m_{y_k^*}(A_i, A_j)$ to be computed as follows:

$$m_{y_k^*}(A_i, A_j) = \begin{cases} 0, \ \textit{if}\ y_k^*(A_i) > 0 \wedge y_k^*(A_j) \le 0 \\ 1, \ \textit{if}\ y_k^*(A_i) \le 0 \wedge y_k^*(A_j) > 0 \\ \dfrac{\left(d_k - y_k^*(A_i)\right)}{\left(d_k - y_k^*(A_i)\right) + \left(d_k - y_k^*(A_j)\right)}, \\ \quad \textit{if}\ y_k^*(A_i) \le 0 \wedge y_k^*(A_j) \le 0 \\ 1 - \dfrac{\left(y_k^*(A_i) - d_k\right)}{\left(y_k^*(A_i) - d_k\right) + \left(y_k^*(A_j) - d_k\right)}, \\ \quad \textit{if}\ y_k^*(A_i) \ge 0 \wedge y_k^*(A_j) \ge 0 \end{cases} \qquad (44)$$

Here, the small positive value d_k guarantees that the denominator in the expression for $m_{y_k^*}(A_i, A_j)$ will not be equal to zero. The membership function $m_{y_k^*}(A_i, A_j)$ shows how "good" the alternative A_i is compared to the alternative A_j with respect to satisfaction of the *k*-th inequality constraint of type (18). The third and fourth expressions in (44) allow to compare A_i with A_j depending on the distance of $y_k^*(A)$ from the boundary (zero) of the inequality constraint. Thus, if both alternatives satisfy this constraint, it is accepted that the better alternative is the one whose value $y_k^*(A)$ is more far from constraint boundary and respectively, if both alternatives don't satisfy this constraint, the better alternative is the one whose value $y_k^*(A)$ is more close to the boundary. The membership function of the total preference relation Q_y^* with respect to all constraints (18) is computed as follows:

$$m_{y^*}(A_i, A_j) = \sum_{k=1}^{l_1^*} l_{y_k^*} \cdot m_{y_k^*}(A_i, A_j), \qquad (45)$$

where $l_{y_k^*}$ is the weight of the k-th constraint. These weight coefficients should satisfy:

$$\sum_{k=1}^{l_1^*} l_{y_k^*} = 1, \quad l_{y_k^*} > 0 \qquad (46)$$

3.3. Determine the fuzzy preference relation with respect to the equality constraints to be satisfied. Similarly, l_2^* individual preference relations $Q_{j_k^*}$ correcponding to the l_2^* equality constraints (19) to be satisfied, are considered:

$$Q_{j_k^*} = \{(A_i, A_j) : A_i, A_j \in A,\ m_{j_k^*}(A_i, A_j)\} \\ k = 1,2,\ldots,l_2^* \qquad (47)$$

where it is proposed for the membership function $m_{j_k^*}(A_i, A_j)$ to be computed as follows:

$$m_{j_k^*}(A_i, A_j) = \begin{cases} 0, \ \textbf{\textit{if}}\ \{j_k^*(A_i) > d_k \vee j_k^*(A_i) < -d_k\} \\ \quad \wedge -d_k \le j_k^*(A_j) \le d_k \\ 1, \ \textbf{\textit{if}}\ -d_k \le j_k^*(A_i) \le d_k \wedge \\ \quad \{j_k^*(A_j) > d_k \vee j_k^*(A_j) < -d_k\} \\ 1 - \dfrac{\left|j_k^*(A_i)\right|}{2 + \left|j_k^*(A_i)\right| + \left|j_k^*(A_j)\right|}, \\ \quad \textbf{\textit{if}}\ -d_k \le j_k^*(A_i) \le d_k \\ \quad \wedge -d_k \le j_k^*(A_j) \le d_k \\ 1 - \dfrac{\left|j_k^*(A_i)\right|}{\left|j_k^*(A_i)\right| + \left|j_k^*(A_j)\right|}, \\ \quad \textbf{\textit{if}}\ \{j_k^*(A_i) > d_k \vee j_k^*(A_i) < -d_k\} \\ \quad \wedge \{j_k^*(A_j) > d_k \vee j_k^*(A_j) < -d_k\} \end{cases} \qquad (48)$$

Here, the small positive value d_k is used to relax the equality constraints (19). The third expression in (48) guarantees that $m_{j_k^*}(A_i, A_j) = m_{j_k^*}(A_j, A_i) = 0.5$ when $j_k^*(A_i) = j_k^*(A_j) = 0$. The membership function $m_{j_k^*}(A_i, A_j)$ shows how "good" the alternative A_i is compared to the alternative A_j with respect to satisfaction of the k-th equality constraint of type (19). The membership function of the total preference relation Q_{j^*} with respect to all constraints (19) is computed as follows:

$$m_{j^*}(A_i, A_j) = \sum_{k=1}^{l_2^*} l_{j_k^*} \cdot m_{j_k^*}(A_i, A_j), \qquad (49)$$

where $l_{j_k^*}$ is the weight of the k-th constraint. These weight coefficients should satisfy:

$$\sum_{k=1}^{l_2^*} l_{j_k^*} = 1, \quad l_{j_k^*} > 0 \qquad (50)$$

3.4. Determine the total fuzzy preference relation and the fuzzy subset of nondominated alternatives.

The total fuzzy preference relation is defined as:

$$Q_T = \{(A_i, A_j) : A_i, A_j \in A,\ m_T(A_i, A_j)\} \qquad (51)$$

and it shows how "good" the alternative A_i is compared to the alternative A_j with respect to optimization of the multiple criteria (4), the satisfaction of the inequality constraints of type (18) and the satisfaction of the equality constraints of type (19). The membership function $m_T(A_i, A_j)$ is computed as follows:

$$m_T(A_i, A_j) = l_I \cdot m_I(A_i, A_j) + l_{y^*} \cdot m_{y^*}(A_i, A_j) \\ + l_{j^*} \cdot m_{j^*}(A_i, A_j) \qquad ,(52)$$

where the weight coefficients l_I, l_{y^*} and l_{j^*} reflect the importance of the optimization criteria compared to that of the inequality and the equality constraints. It is evident that:

$$m_T(A_i, A_j) + m_T(A_j, A_i) = 1 \qquad (53)$$

Let $m_T^s(A_i, A_j)$ be the corresponding strictly fuzzy preference realtion to $m_T(A_i, A_j)$ with the membership function [3]:

$$m_T^s(A_i, A_j) = \max\{[m_T(A_i, A_j) - m_T(A_j, A_i)], 0\} \qquad (54)$$

Then the fuzzy subset of nondominated alternatives is described with a membership function as [4]:

$$m_T^{n.d.}(A_i) = 1 - \max_{A_j \in A} m_T^s(A_j, A_i) \qquad (55)$$

3.5. Select the best alternative A_{best}.

The best alternative is the one that maximizes $m_T^{n.d.}(A_i)$ [4], i.e.:

$$A_{best} = \arg \max_{A_i \in A} m_T^{n.d.}(A_i) \qquad (56)$$

4. Results from the solution of controller synthesis problem.

The results of performing steps 1 through 3 is that the best alternative A_{best} of controller structure is selected (in step 3) with the corresponding optimal values of controller parameters $\underline{p}_{opt}(A_{best})$ (determined in step 2).

4. Approach for selection of weight coefficients.

Here, the approach proposed in [5] is accepted to select the weight coefficients. In [5] this method has been applied to the problem of determination of weight coefficients of the uncertain parameters when solving optimization problems under parameter uncertainty. Here, it is shown that this method can be successfully used in the solution of controller synthesis problem described in this paper. In section 2 several weight coefficients were introduced:

$$\begin{aligned} & l_{I_k},\ \boldsymbol{k}=1,2,\dots,\boldsymbol{m} \\ & l_{y_k^*},\boldsymbol{k}=1,2,\dots,\boldsymbol{l}_1^* \\ & l_{j_k^*},\ \boldsymbol{k}=1,2,\dots,\boldsymbol{l}_2^* \\ & l_I,\ l_{y^*},\ l_{j^*} \end{aligned} \qquad (57)$$

As an example, the approach in [5] will be applied to determine the weight coefficients l_{I_k}, $\boldsymbol{k}=1,2,\dots,\boldsymbol{m}$ of the optimality criteria (4). The following fuzzy preference relation is considered:

$$\boldsymbol{Q}=\{(\boldsymbol{I}_j,\boldsymbol{I}_k):\boldsymbol{I}_j,\boldsymbol{I}_k\in\{\boldsymbol{I}_1,\boldsymbol{I}_2,\dots\boldsymbol{I}_m\},\ m(\boldsymbol{I}_j,\boldsymbol{I}_k)\}\ ,(58)$$

where $\boldsymbol{I}_k$, $\boldsymbol{k}=1,2,\dots,\boldsymbol{m}$ are the optimality criteria (4) to be optimized. The membership function $m(\boldsymbol{I}_j,\boldsymbol{I}_k)\}$ can take only three values and is determined as follows:

$$m(\boldsymbol{I}_j,\boldsymbol{I}_k)=\boldsymbol{a}_{jk}=\begin{cases}1,\ \boldsymbol{if}\ \ \boldsymbol{I}_j\succ\boldsymbol{I}_k\\ 0,\ \boldsymbol{if}\ \ \boldsymbol{I}_j\prec\boldsymbol{I}_k\\ 0.5,\ \boldsymbol{if}\ \ \boldsymbol{I}_j\leftrightarrow\boldsymbol{I}_k\end{cases} \qquad (59)$$

In (59) the symbol "$\succ$" means that the optimality criterion $\boldsymbol{I}_j$ is more important than criterion $\boldsymbol{I}_k$, the symbol "$\prec$" means that $\boldsymbol{I}_j$ is less important than $\boldsymbol{I}_k$ and the symbol $\leftrightarrow$ is used to denote that $\boldsymbol{I}_j$ and $\boldsymbol{I}_k$ are equally important. Then the table with the values $\boldsymbol{a}_{jk}$ of the membership function (59) is constructed.

It is accepted that:

$$\boldsymbol{a}_{jj}=1,\ \ \boldsymbol{j}=1,2,\dots,\boldsymbol{m} \qquad (60)$$

In Table 1, the coefficients $\boldsymbol{S_j}$ are computed as:

$$\boldsymbol{S}_j=\sum_{\boldsymbol{k}=1}^{\boldsymbol{m}}\boldsymbol{a}_{jk}-1 \qquad (61)$$

Then, the weight coefficients l_{I_j} are determined as follows:

$$l_{I_j}=\frac{\boldsymbol{S}_j}{\sum_{\boldsymbol{k}=1}^{\boldsymbol{m}}\boldsymbol{S}_k} \qquad (62)$$

and they satisfy:

$$\sum_{\boldsymbol{j}=1}^{\boldsymbol{m}} l_{I_j}=1 \qquad (63)$$

The other weight coefficients in (57) can be determined in a similar way.

Table 1. Values of the membership function of the fuzzy preference relation (58).

$\boldsymbol{I_k}\rightarrow$ $\boldsymbol{I_j}\downarrow$	$\boldsymbol{I_1}$	$\boldsymbol{I_2}$	…	$\boldsymbol{I_k}$	…	$\boldsymbol{I_m}$	$\boldsymbol{S_i}$	l_{I_j}
$\boldsymbol{I_1}$	$\boldsymbol{a_{11}}$	$\boldsymbol{a_{12}}$	…	$\boldsymbol{a_{1k}}$	…	$\boldsymbol{a_{1m}}$	$\boldsymbol{S_1}$	l_{I_1}
$\boldsymbol{I_2}$	$\boldsymbol{a_{21}}$	$\boldsymbol{a_{22}}$	…	$\boldsymbol{a_{2k}}$	…	$\boldsymbol{a_{2m}}$	$\boldsymbol{S_2}$	l_{I_2}
⋮	⋮	⋮	…	⋮	…	⋮	⋮	⋮
$\boldsymbol{I_j}$	$\boldsymbol{a_{j1}}$	$\boldsymbol{a_{j2}}$	…	$\boldsymbol{a_{jk}}$	…	$\boldsymbol{a_{jm}}$	$\boldsymbol{S_j}$	l_{I_j}
⋮	⋮	⋮	…	⋮	…	⋮	⋮	⋮
$\boldsymbol{I_m}$	$\boldsymbol{a_{m1}}$	$\boldsymbol{a_{m2}}$	…	$\boldsymbol{a_{mk}}$	…	$\boldsymbol{a_{mm}}$	$\boldsymbol{S_m}$	l_{I_m}

References

[1]. S. Engell, R. Muller, "Fast and efficient selection of control structures", *Proceedings of ESCAPE-1*, Elsinore, Denmark, 24-28 May, 1992, pp.157-164.

[2]. G. Sullivan, "Development of feed changeover policies for refinery distillation units", *Ph.D. thesis*, University of London, London, England, 1977.

[3]. I. Popchev, V. Peneva, "An algorithm for comparison of fuzzy sets", *Fuzzy Sets and Systems*, vol.60, 1993, pp.59-65.

[4]. S. Orlovski, "Decision making with a fuzzy preference relation", *Fuzzy Sets and Systems*, vol.1, 1978, pp.155-167.

[5]. R. Pavlova, "Optimization of technological systems under uncertainty", *Ph.D. thesis* (In Bulgarian), University of Chemical Technology and Metallurgy, Sofia, Bulgaria, 1988.