# On the Design of Time-Optimal Sliding Mode Control

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*Abstract*: - We demonstrate a time-optimal control algorithm based on the sliding mode control principle. A designed time-optimal trajectory during the reaching phase is combined with fast sliding dynamics. The discontinuous algorithm gives time response, closer to the analytical time-optimal control solution based on Pontryagin principle, and robust performance to plant parameter uncertainties. Conditions for the existing of time-optimal sliding mode trajectory for second order systems are derived. A fuzzy version is also demonstrated.

*Keywords*: - time-optimal control, sliding mode control, fuzzy time-optimal control
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## 1. Introduction

The time-optimal control (TOC) is important for robots and manipulators, where a given trajectory has to be passed in the shortest possible time with or without constraints on some state coordinates like velocities, accelerations, etc. If the robot has invariant dynamics the time-optimal solution can be obtained analytically. But in reality the robot continuously changes its kinematic and dynamic properties and needs at least adaptive control leading to quite complex algorithms.

Alternatively we can use the simple sliding mode control (SMC) algorithm already employed successfully to many robot applications [1, 2, 3]. The sliding mode keeps good system properties. On other hand it does not guarantee TOC performance, so one has to compromise between achieved robustness and longer response time.

In this paper we try to bridge both TOC and SMC principles and show how SMC can give robust and quasi-TOC behavior if the theoretical switching line for a second order robot dynamics is used as sliding line in a SMC. For better understanding of the main idea we introduce a very simple example. Let first have as a system under control a double integrator described by equation (14). Obviously this is the simplest second order system with the simplest switching line (15) for TOC [4] giving the behavior on Fig. 1.

Now suppose the plant has realistic dynamics with two real poles and the TOC is using the same switching line. To emphasize on the resulting effect let examine poles at $\{\tilde{n}_1 = -0.01, \tilde{n}_2 = -0.02\}$, $\{\tilde{n}_1 = -0.1, \tilde{n}_2 = -0.2\}$ and $\{\tilde{n}_1 = -1, \tilde{n}_2 = -2\}$.Let also amplify the relay output by

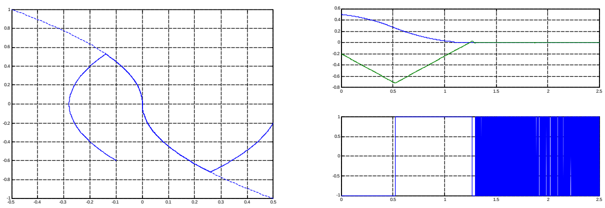


Fig. 1. Time-optimal behavior of double integrator

additional gain $K_{ad} = (\tilde{n}_1 \tilde{n}_2)^{-1}$. The simulated behavior with the same initial conditions is on Fig. 2. The settling time of the examples is almost the same, even the time constants differ $10^2$ times. This seems to be a paradox, but it is due to the high gain in the system. Extending the experiment by inserting extremely high gain, let say $10^6$, the system will perform SMC on the TOC switching line of the double integrator. In other words we are able to replicate the TOC double integrator behavior regardless the dynamics of the real plant. If the poles are shifted left the settling time becomes smaller as the plant time constants are decreasing.

From this example we can theoreticaly conclude that a robust TOC behavior of a second order system, when the poles vary during system motion, can be achieved by introducing sufficiently large gain. Hence, if the gain of the plant under control is smaller we have to compensate it to achieve TOC and SMC conditions. The problem is related to the design of a SMC in both standard and fuzzy versions.

Guidelines for design and implementation of this approach are given in Section 4. Section 2 gives the problem formulation, then Section 3 contains a brief overview of TOC and SMC. Illustrative examples are shown in Section 5. Final comments conclude the work.

## 2. Problem Formulation

Let have a controllable and reachable second order time-invariant linear plant

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \mathbf{A}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \mathbf{B}u(t) + \mathbf{f}(\mathbf{x},u,t) + \mathbf{d}(t) \ , \qquad (1)$$

where $\mathbf{d}(t)$ is a bounded disturbance and the bounded by some known function term $\mathbf{f}(\mathbf{x},u,t)$ represents uncertainties in the model. The control signal is limited to certain maximum values, for example the normalized value $|u| \le 1$. We suppose also the classical time-optimal problem

$$I(\mathbf{x},\mathbf{u}) = \int_{t_0=0}^{t_f} 1.dt \Rightarrow \min \quad . \qquad (2)$$

can be solved for the undisturbed and certain plant model

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \mathbf{A}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \mathbf{B}u(t) \qquad (3)$$

and the solution is unique under respective conditions.

Naturally uncertainties exist, so parameter variations (poles move) in (3) can make the system behavior quite different from the optimal, normally designed for nominal parameters. In addition disturbances can also deteriorate the system performance far away from the expected. The problem is to develop an algorithm for time-optimal control of plants (1) accomodating robust properties and providing time-optimal performance closer to the theoreticaly achievable from (2) as for the pure case (3).

The task can be solved by combining the positive characteristics of two well known methods that are different in nature but similar in the way the control action is applied to the plant. The SMC provides robustness and partly satisfactory time behavior but just for the sliding mode. The motion to the sliding surface does not fulfil minimum time requirement. On other side the TOC approach gives the desired solution, but the control algorithm is not robust. We propose to merge both approaches into an algorithm utilizing some common details, namely the switching or sliding surface (line).

## 3. Short Overview of Time-Optimal and Sliding Mode Control

### 3.1. Time-optimal control

According to the Pontryagin principle the Hamiltonian for the problem (2) and plant (3) is given by

$$\mathrm{H}(\mathbf{x},\mathbf{u},\ddot{\mathbf{e}}) = 1 + \hat{\mathbf{a}}^T(t)\mathbf{A}\mathbf{x}(t) + \hat{\mathbf{a}}^T(t)\mathbf{B}\mathbf{u}(t) \ , \qquad (4)$$

where $\hat{\mathbf{a}}(t)$ is Lagrange multiplier obtained by solving

$$\dot{\hat{\mathbf{a}}}(t) = -\left[\frac{\partial \mathrm{H}(\mathbf{x},\mathbf{u},\hat{\mathbf{a}})}{\partial \mathbf{x}(t)}\right]^T = -\mathbf{A}^T\hat{\mathbf{a}}(t), \quad t \le t_f \ . \qquad (5)$$

The minimum of $\mathrm{H}(\mathbf{x},\mathbf{u},\hat{\mathbf{a}})$ under the constraints $|u| \le 1$ is obtained by a discontinuous optimal control

$$\mathbf{u}_{opt}(t) = -\mathrm{sgn}[\mathbf{B}^T\hat{\mathbf{a}}(t_0)] \ . \qquad (6)$$

In the general case the relation of $\hat{\mathbf{a}}(t_0)$ from $\mathbf{x}(t_0)$ is nonlinear and to obtain the optimal control signal as a function of time is impossible. To avoid the necessity to solve the differential equation (5) about $\hat{\mathbf{a}}(t)$ a nonlinear state feedback can be implemented, so

$$\mathbf{u}_{opt}(t) = -\mathrm{sgn}[\mathbf{K}(\mathbf{x}_{opt}(t))] \ , \qquad (7)$$

where $\mathbf{x}_{opt}(t)$ is the optimal motion. It is necessary:

(i) the plant has to be stable (no positive poles);
(ii) the control variables switch at most $n-1$ times, where $n$ is the number of the poles;
(iii) if exists, the solution is unique for any fixed $\mathbf{x}(t_f)$;

in order to derive time-optimal control. The theoretical forms of the nonlinear state feedback (7) of second order systems are provided in the examples in Section 5.

### 3.2. Sliding Mode Control

For the real controllable plant (1) a family of switching hyperplanes

$$S = \{\ \mathbf{x} \mid s_i(\mathbf{x}) = 0\ \} \ , \qquad (8)$$

can be defined, where $s_i(\mathbf{x})$ is a switching function. The system motion from any initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ to the phase space origin consists of two components - reaching mode and sliding mode. During the reaching mode the system moves towards the switching surface in finite time. When certain conditions for existing of sliding mode are satisfied [1], the system begins to slide on a switching hyperplane until reaches the origin. The general condition for sliding is $s_i\dot{s}_i < 0$ in the neighborhood of $s_i(\mathbf{x})$ [6]. When $s_i(\mathbf{x}) = 0$ and $\dot{s}_i(\mathbf{x}) = 0$ the system is independent on system parameter variations and disturbances.

The sliding motion depends highly on (8) and in the general case takes infinite time. Choosing the values of the sliding hyperplane parameters one can obtain faster dynamics and reduce the sliding mode transient time.

## 4. Time-Optimal Sliding Mode Control

Let have a look at both parts of the sliding mode control. Both reaching and sliding modes are executed in sequence. If we make sure that separately during the reaching mode and the sliding mode the algorithm comes up with a minimal transient time, then the total response time will be also minimal. To show the consistency of this hypothesis we have to investigate how to design:

(1) minimum-time control for the sliding mode,
(2) minimum-time control for the reaching mode.

The first question is addressed to the variable structure control theory and probably is easier, while the second one is addressed also to the time-optimal control theory and needs a synchronised view in both theoretical areas.

### 4.1. Minimum-time control in sliding mode

As the system dynamics depends only on the dynamic equation of the sliding hyperplanes (8), proper parameter selection could give satisfactory behavior. Let consider *n*-th order SISO system when equation (8) takes the form

$$s(\mathbf{x},t) = (\frac{d}{dt} + l)^{n-1} x(t) = \sum_{k=0}^{n-1} \binom{n-1}{k} x_{n-1-k}(t) \qquad (9)$$

The positive parameter $l$ together with (9) describes the dynamics in sliding mode. Physically $l$ can be treated as a time constant in a chain of first order blocks, which is always stable. Hence, for second order system we get the simplest case of straight sliding line

$$x_2(s) + l x_1(s) = s.x_1(s) + l x_1(s) = s + l = 0 \quad .$$

Increasing $l$ the pole goes to infinity and the sliding system response can be made short enough. However, increasing $l$ the domain of attraction for sliding mode drastically shrinks [1] that probably introduces undesirable oscillatory response. Therefore some upper limit of $l$ should be taken into account if pure SMC algorithm is designed. Concluding we underline that small but sufficient domain of attraction for sliding mode exists and at the same time it provides fast enough system time response. Alternatively it is possible to set a high gain and then the closed loop system follows the behavior, corresponding to the switching (sliding) line.

### 4.2. Minimum-time control in reaching mode

If the plant has the properties described at the beginning of Section 2, the reaching motion can be provided by a linear state feedback. Obviously the time to enter the sliding mode domain of attraction depends on the initial condition and the location of the switching (sliding) hyperplanes. As mentioned above a small domain can introduce many switchings (bang-bang control signal) and the response time will be increased along with some undesired oscillations.

Then a reasonable questions is: is it possible to employ the time-optimal control in terms of the classical Pontryagin minimum principle for the reaching mode? In this way we should guarantee strict conditions for minimum-time performance of the reaching mode. The reply we introduce in intuitive but obvious way. In parallel we also will try to bypass the most difficult problem in the Pontryagin's approach – the design of switching line that could be of higher complexity. The main idea here is:

(i) replace the exact theoretical switching line for TOC with a switching line of a simple second order plant, for example double integrator;
(ii) make a linearwise approximation of it as simple as possible or apply fuzzy approximation, if fuzzy logic controller will be used;
(iii) use the simplified switching line to bring the system to the sliding mode domain of attraction, then slide to the origin.

### 4.3. Design algorithm

The rationale behind the proposed approach consists of the following.

We can define a desired time-optimal behavior of a second order system assuming some kind of normalized properties. For example this can be a system with unit gain $K_\Sigma = 1$, accepting bang-bang input control signal with unit amplitude $|u| \le 1$. Appropriate choice is the simplest system of that kind - the double integrator (see Section 5). Its TOC behavior is easily computed and the properties are transparent from robot control viewpoint. For the precise system model no overshoot response guarantees no oscillations during the system motion.

In practice the system model is imperfect. The poles can have varying locations depending on the robot kinematics. However, we want to keep the system properties closer to the described above. To derive the necessary conditions let consider the plant poles together with their uncertainties

$$\begin{aligned} ñ_1 &= ñ_1^n + \Delta ñ_1 \,, \\ ñ_2 &= ñ_2^n + \Delta ñ_2 \,, \end{aligned} \qquad (10)$$

where $ñ_1^n$, $ñ_2^n$ are the nominal poles and $|\Delta ñ_1| \le d_1$, $|\Delta ñ_2| \le d_2$ are bounded pole variations. Let define the gain parameter of the plant $K_P$ as a product of the poles

$$K_P = \tilde{n}_1.\tilde{n}_2 = K_P^n + \Delta K \quad , \tag{11}$$

where $K_P^n = \tilde{n}_1^n.\tilde{n}_2^n$ and $\Delta K = h(d_1, d_2)$ will be also bounded by $|\Delta K| \le d$ . The value $K_P$ generally would differ to $K_\Sigma$. It is not difficult to see that it could be much smaller than $K_\Sigma$ when the plant time constants (the reciprocial values to the poles) have significant size. Therefore to compensate $K_P$ to $K_\Sigma$ we need additional gain

$$K_{ad} = \frac{K_\Sigma}{K_P} = \frac{1}{K_P} = (\tilde{n}_1\tilde{n}_2)^{-1}. \tag{12}$$

We have to ensure a minimum value of $K_{ad}$ to keep the property (12). As $K_P$ is in the denominator the minimum value $K_P^{\min}$ is important. Hence, for the existing of time-optimal sliding mode trajectory of the system we can formulate the following **condition**:

$$K_{ad} \ge K_P^{\min} = (\tilde{n}_1\tilde{n}_2)_{\min}^{-1} \quad . \tag{13}$$

Under that condition the system gain $K_\Sigma$ will be at least equal to 1 and the closed loop system performance will be not worse than the time-optimal response of the double integrator, considered as reference.

If condition (13) is not satisfied the closed loop system will have longer response time and the requirements for minimum-time control are not guaranteed. On contrary, condition (13) is naturally satisfied for plants with small time constants, giving higher values of $K_P$, hence their responses have shorter settling time.

The time-optimal control strategy requires larger values of the control signal. In such way the system trajectory cannot be disrupted by any bounded and less powerful disturbance. Once started the sliding mode provides the robustness to disturbances.

## 5. Test Examples

We provide a set of illustrative simulations using some well-known examples:

(a) time-optimal control of double integrator with its theoretical switching line (15),

(b) time-optimal control of two-pole plant with its theoretical switching line (17),

(c) time-optimal control of two-pole plant with the switching line of the double integrator (15),

(d) fuzzy SMC of two-pole plant with the approximated switching line of the double integrator (15).

The SISO plant models are:

(1) double integrator

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{14}$$

with theoretical TOC switching line

$$x_1(t) = -0.5 x_2(t) |x_2(t)| \quad . \tag{15}$$

The behavior is already shown on Fig. 1.

(2) two-pole system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\tilde{n}_1\tilde{n}_2 & \tilde{n}_1 + \tilde{n}_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K \end{bmatrix} u(t) \tag{16}$$

where $K = 1$ and poles vary $\tilde{n}_1 = (-1) \div (-0.01)$ and $\tilde{n}_2 = (-2) \div (-0.02)$ . The switching line is

$$y_2(t) = \frac{y_1(t)}{|y_1(t)|}[(1 + |y_1(t)|)^a - 1] \ , \ a = \frac{\tilde{n}_2}{\tilde{n}_1} \tag{17}$$

and $y_1(t), y_2(t)$ are new system coordinates obtained by the transformation [4]

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \frac{1}{K} \begin{bmatrix} -\tilde{n}_1\tilde{n}_2 & \tilde{n}_1 \\ -\tilde{n}_1\tilde{n}_2 & \tilde{n}_2 \end{bmatrix} . \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

for convenience of designing the switching line.

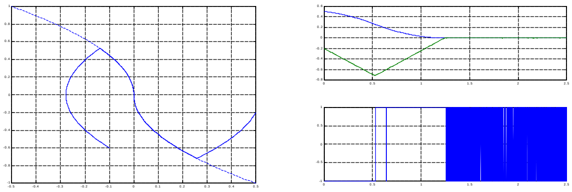
To identify the properties of the proposed algorithm we make several pairwise comparisons and answer some questions:

(a) *Does the switching line of the double integrator give minimum time response for the two real pole system?*

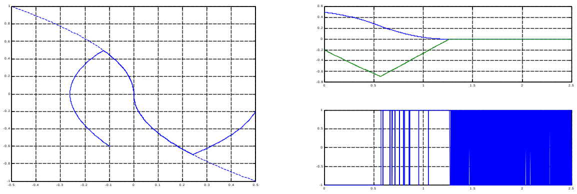
We already pointed out there are slight deviations between Fig. 2 and the double integrator time-optimal behavior on Fig. 1. Comparing Fig.3 and Fig. 2 (c) with the same plant pole locations we observe no difference in the duration of time responses. The difference is in the mode of control. The theoretical TOC needs only one switching and then the chattering keeps the zero steady state, while the sliding control provides a sliding mode immediatelly after the first switching. The chattering in this case in non-uniform because of the nonlinear nature of the switching line. Hence, the conclusion is the time-optimal SMC comes up with a behavior that is not worse than the real time-optimal algorithm.

(b) *How sensitive is the proposed algorithm to variations in the plant model parameters (poles)?*

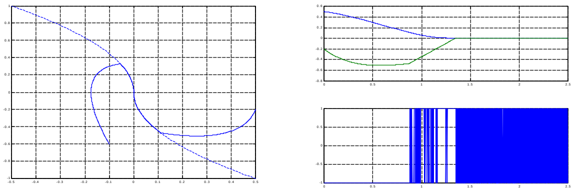
Comparing again Fig.3 and all cases on Fig.2 we observe also slight dissimilarities in the response time duration even the poles differ $10^2$ times. In all cases the



(a) $\{\tilde{n}_1 = -0.01, \tilde{n}_2 = -0.02\}$



(b) $\{\tilde{n}_1 = -0.1, \tilde{n}_2 = -0.2\}$



(c) $\{\tilde{n}_1 = -1, \tilde{n}_2 = -2\}$

Fig. 2. Two-pole system with TOC switching line of the double integrator

transient responses keep the non-overshoot nature of the system output which is a very good property for robot/ manipulator control.

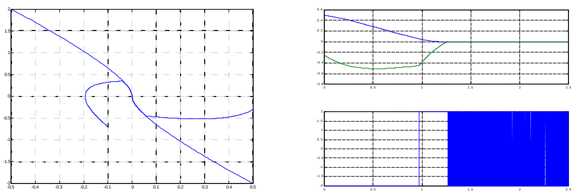


Fig. 3. Two-pole system with its theoretical TOC switching line

(c) *How a fuzzy controller can achieve a time-optimal control behavior?*

A standard two-input-one-output fuzzy controller which approximates the theoretical switching line of the two-pole $\{\tilde{n}_1 = -1, \tilde{n}_2 = -2\}$ system is used to control it. Both inputs receive the state coordinates $x_1(t), x_2(t)$ and the output is the control variable sent to the switch. The behavior is presented on Fig. 4. Obviously, there are no visual differences with the respective crisp time-optimal algorithm, as expected. The fuzzy membership functions and control surface are presented on Fig. 6 for illustration. Then the approximated swithing line of the double integrator is inserted in the same fuzzy controller. The behavior in the control of the two-pole $\{\tilde{n}_1 = -1, \tilde{n}_2 = -2\}$ system is shown on Fig. 6. Again very small differences with the respective crisp equivalent on Fig. 2 (c) or the TOC case on Fig. 4. Hence, the conclusion is that the fuzzy version of the proposed algorithm achieves the time-optimal control property to the same level as the crisp original.

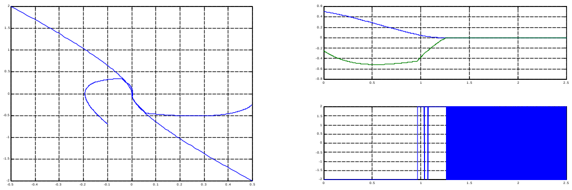


Fig. 4. Two-pole system $\{\tilde{n}_1 = -1, \tilde{n}_2 = -2\}$ with its fuzzy approximated TOC switching line

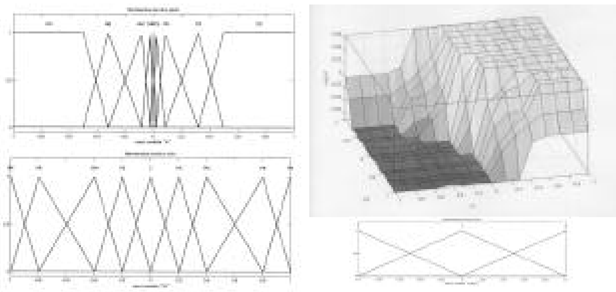


Fig. 5. Membership functions and control surface for TOC line of the two-pole system $\{\tilde{n}_1 = -1, \tilde{n}_2 = -2\}$

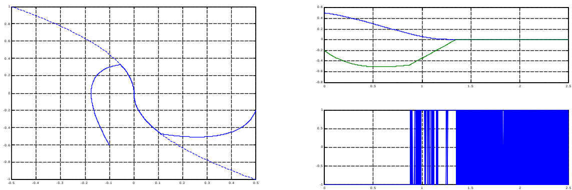


Fig. 6. Two-pole system $\{\tilde{n}_1 = -1, \tilde{n}_2 = -2\}$ with fuzzy approximated TOC switching line of the double integrator

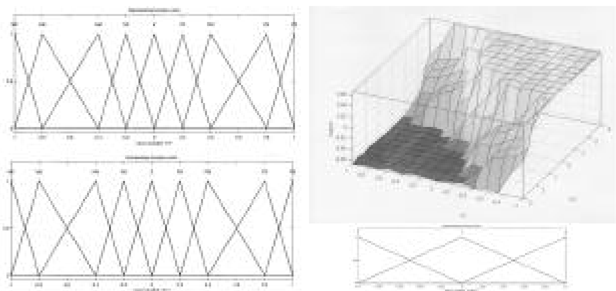


Fig. 7. Membership functions and control surface for TOC line of the double integrator

## 6. Conclusions

A simple approach to design a time-optimal sliding mode control algorithm is presented. It is based on the conclusions about the time-optimal behavior of a double integrator and is applicable for second order systems. It is convenient for implementation and has also good robust properties. The provided test examples show these properties for SISO systems like manipulators/ robots. A particular interest is the achieved non-overshoot time response which is very useful in positioning for example of moving robot arms.

Open questions is related to plants with time delay or to the problem arising when higher order system is approximated with lower order with time delay. Possible solution could be found applying Smith-predictor or internal model control techniques to compensate the delay and then ground the proposed algorithm on the compensated system model.

Obvious drawbacks of the proposed algorithms are the large control signal during the reaching mode and the chattering in the sliding mode. Certain known precautions not discussed here can be applied slightly affecting the properties of the algorithm.

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