Some Results on Observability of Polynomial Systems With First Integrals

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Abstract: - The observability problem for a polynomial system possessing a polynomial first integral is examined. The property of distinguishability of level sets of a first integral is introduced. It is demonstrated that in the generic situation the polynomial system has this property everywhere in the state space excepting, may be, some subset of the equilibria set. Further, the property of strong distinguishability of level sets of a first integral is also introduced and one sufficient condition is given. It is discussed how to use the latter property together with the parametrized set of Luenberger-type observers for state estimation. Finally, we concern observability analysis on a level set and give some necessary and sufficient conditions.

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1 Introduction

Nonlinear systems possessing first integrals form an important class of systems due to applications in mechanics, physics, chemistry etc. However, the topic of observability for this class of systems is a weakly elaborated problem in control theory. One related reference is [1].

The main goal of this paper is to describe observability conditions for nonlinear systems with first integrals. We consider a polynomial system

$$dx/dt=f(x), \quad x\in \mathbf{R}^n \qquad (1)$$
$$y=h(x), \quad y\in \mathbf{R}^1$$

where h is a polynomial observation law. Let $j(x,t)$ be the solution of f with the initial condition $j(x,0)=x$. Suppose that the system (1) has a polynomial first integral $r\,{}^{1}0; \quad r:\mathbf{R}^n\rightarrow\mathbf{R}^1$.

Definition 1. Two states $x^1;x^2$ are called distinguishable if there is an instant $t\geq 0$ such that $h(j(x^1,t))\neq h(j(x^2,t))$. Let M be some invariant set of the phase flow of the system (1). The system (1) is called observable on the invariant set M if any pair of distinct states from M is distinguishable.

Definition 2. The system (1) has distinguishable level sets of the first integral r with respect to the invariant set M if for any $a^1;a^2\in r(M), \quad a^1\,{}^{1}a^2$, we have: any states $x^s\in r^{-1}(a^s)\cap M, \quad s=1,2$, are distinguishable .

Based on algebraic ideas taken from [2] we establish conditions under which the system (1) has distinguishable level sets of the first integral r. Then we concern the theoretical procedure for obtaining estimates of states based on another concept of distinguishable level sets, normal forms and Luenberger-type observers.

Finally, we concern observability analysis on a level set and give some necessary and sufficient conditions.

2 Some preliminaries

Let $L_f h$ be a Lie derivative of the function h along the vector field f and let $L_f^s h=L_f(L_f^{s-1}h), \ s\geq 2; \ L_f^0 h:=h.$

We introduce the mapping $H_m(x)=(h(x),L_f h(x),\ldots,L_f^{m-1}h(x))$. By $Alg(f,h)$ we denote the algebra of polynomials of n real variables $x_1,\ldots,x_n$, which is formed with help of $\{L_f^s h; \ s=0,1,\ldots\}$ We define a linear vector space $V(r;n,d)$ of polynomial vector fields f corresponding to (1) such that 1) r is their common first integral; 2) the degree $deg f_s\leq d; \ s=1,2,\ldots,n.$ Also, we define a linear vector space $OL(n,l)$ of scalar polynomial observations laws $h:\mathbf{R}^n ® \mathbf{R}^1, \ \ deg h\leq l.$

3 Main results

Below we describe conditions under which states of distinct level sets of r generates distinct outputs. Then we reduce the state estimation problem posed for states of $\mathbf{R}^n$ to the state estimation problem for states of some level set.

We establish

Theorem 1. There exists an open, dense and semialgebraic set $W\subset V(r;n,d)\times OL(n,l)$ such that if $(f,h)\in W$ then

1) the set $f^{-1}(0)$ is finite and there is a subset $O\subset f^{-1}(0)$ such that the system (1) has distinguishable level sets of the first integral r with respect to the set $M=\boldsymbol{R}^n\backslash O$.
2) Moreover, one can find real polynomials of $n+1$ variables p and q for which $r(x)=p(H_{n+1}(x))/q(H_{n+1}(x))$, provided $q(H_{n+1}(x))\neq 0$.

Sketch of the proof. Let $r\notin Alg(f,h)$ otherwise the assertion is trivially true with $O=\emptyset$. Assume that $L_f^{n-1}h$ is a non-constant polynomial. By use of the Perron theorem on algebraic dependence, [3], to polynomials $L_f^s h$; $s = 0,1,\ldots, n-1$; r; we deduce that there is a nontrivial polynomial Φ of $n + 1$ real variables for which $\Phi(H_n(x),\ r(x))\equiv 0$. We can rewrite this identity in the form

$$S^d_{s=0}\,[r(x)]^s a_{s0}(H_n(x)) \equiv 0 \qquad (2)$$

for some integer $d\geq 1$ and some real polynomials a_{s0} ; $s = 0,\ldots, d$. Now we apply the Lie derivative L_f to the identity (2). It is easy to see that we come to another polynomial identity

$$S^d_{s=0}\,[r(x)]^s a_{s1}(H_{n+1}(x)) \equiv 0 \qquad (3)$$

Then we use the following lemma, [6].

Lemma 2. Suppose that $L_f^{n+1}h$ is a non-constant polynomial. Then there are non-constant polynomials a; b_j; $j = 1, 2,\ldots$, all of $n + 1$ real variables, such that the following sequence of polynomial identities

$$L_f^{n+j}h(x)[a(H_{n+1}(x))]^{\,j}+bj(H_{n+1}(x)) \qquad (4)$$

$j=1,2,\ldots$

is valid.

Now we apply the Lie derivative L_f to (3). As a result, we come to the polynomial identity

$$S^d_{s=0}\,[r(x)]^s b_{s1}(H_{n+2}(x)) \equiv 0 \qquad (5)$$

for some polynomials b_{s1},$s = 0, 1,\ldots, d$, of $n + 2$ variables. Then we take the identity (5). We substitute there instead of $L_f^{n+1}h$ its expression obtained from (4) with $j = 1$. As a result, we come to the polynomial identity

$$S^d_{s=0}\,[r(x)]^s a_{s2}(H_{n+1}(x)) \equiv 0$$

where a_{s2}, $s = 0,\ldots, d$, are some real polynomials of $n + 1$ variables. We repeat this argument for $j = 2, 3,\ldots$, and finally, we have the following system

$$S^d_{s=0}\,[r(x)]^s a_{si}(H_{n+1}(x)) \equiv 0, \quad i=1,\ldots,d-1 \qquad (6)$$

The system (6) is linear with respect to $[r(x)]^s$, $s = 1,\ldots,d-1$. All a_{si}; $s = 0,\ldots, d$; $i = 1,\ldots, d-1$, are polynomials of $n + 1$ variables. Its determinant D_0 is a polynomial depended on the vector of coefficients (coef(f), coef(h)) of the pair (f, h) and x as well.

We substitute into equation D_0(coef(f), coef(h), x) = 0 instead of x the solution $j(x, t)$ and calculate (n - 1) time derivatives at $t = 0$. As a result, we obtain the polynomial system of equations D_s(coef(f), coef(h), x) = 0, $s = 0, 1,\ldots, n-1$.

Now by arguments of the proof of Theorem 3, [2], we come to the first desirable assertion. The second assertion is followed from the solution of the system (6) with respect to r.

Similarly we come to the following assertion:

Theorem 1a. For any polynomial pair (f,h) such that $L_f^{n-1}h$ is a non-constant polynomial there is an open and dense invariant set $M\subset\boldsymbol{R}^n$ such that the restriction $(f,h)|_M$ has distinguishable level sets on M.

In practice it is more convenient to apply one more rough concept of distinguishable level sets than the concept contained in Definition 2.

Below up to the end of Section 3 we consider that $f_1,\ldots,f_n,h,r$ are sufficiently smooth.

Definition 3. The system (1) has strongly distinguishable level sets of the first integral r with respect to the invariant set M if for any $a^1;a^2\in r(M)$, a^1 ¹ a^2, and any two phase curves $Á_s\subset r^{-1}(a^s)$, s=1,2, we have : $h(Á_1)$ ¹ $h(Á_2)$.

Now we formulate one sufficient condition for strongly distinguishability of level sets of the first integral.

Let $h(x_1,\ldots,x_n)=(x_1,\ldots,x_p)^T$, $0_p:=(0,\ldots,0)\in\boldsymbol{R}^p$ and we define the following sets:

1) $L(x)=Cl\cup_t pj(x,t)$; here

$$pj(x,t)=(\,j_1(x,t),\ldots,j_p(x,t));$$

2) for the invariant set $M\subset\boldsymbol{R}^n$ we introduce the set $K(a,M)=\{L(x)\mid x$ Î $M\cap r^{-1}(a)\}$.

Proposition 1. Assume that following conditions are valid:

1) the first integral
$$r(x_1,\ldots,x_n)=r_1(x_1,\ldots,x_p)+r_2(x_{p+1},\ldots,x_n);$$
2) the function $r_2(x_{p+1},\ldots,x_n)$ attains a global, may be non-strict, extremum on $M\cap 0_p\times\boldsymbol{R}^{n-p}$ (we denote its value by α);
3) the closure of any phase curve in M intersects the set
$$\bigcup_{(x_{p+1}^*,\ldots,x_n^*)\in r_2^{-1}(a)} \{x \text{ Î } \boldsymbol{R}^n \mid x_i=x_i^*;\ i=p+1,\ldots,n\}.$$

Then (1) has strongly distinguishable level sets of the first integral r with respect to M.

The proof consists in the easy analysis of the set

$K(a,M)$ and is omitted here. We note also that Proposition 1 is a generalization of one assertion in [1] respecting to the Euler equation of the rigid body dynamics with the linear observation of one coordinate.

In practice, we can determine approximately the level set by the following operations:

1) Observe time functions $\varphi_1(x,t),\ldots,\varphi_p(x,t)$ on some time interval [0,T]; T≤∞;
2) Calculate a global extremum $\delta:=\rho_1(\varphi_1(x,t^*),\ldots,(\varphi_p(x,t^*))$ of the function $Y(t):=\rho_1(\varphi_1(x,t),\ldots,\varphi_p(x,t))$ at some instant $t^*\in[0,T]$;
3) Calculate the parameter a of the level set by the formula $a:=d+\alpha$.

Here we do not discuss the situation when the global extremum of $Y(t)$ is attained outside of the observation interval [0,T].

Example 1. Let

$(dx_1/dt,dx_2/dt)^T=(x_2{}^2,-x_1x_2)^T$,

$h(x_1,x_2)=x_1$; $M=\{ x_2\geq 0 \}$; $\rho=x_1{}^2+x_2{}^2$.

This system is satisfied to conditions of Proposition 1.

4 One remark on state estimation with help of observers

One can see from Sections 3 that the concept of strongly distinguishable level sets is more adapted for the solution of state estimation problem because we can obtain information about a level set directly from the outputs.

Assume that we find the parameter a of the level set corresponding to some output. Let $x^*\in M$. By (A,C) we denote the dual Brunovsky canonical form, with A be the $(n\times n)$-matrix; C be the $(2\times n)$-matrix.

Also, let

$$dz/dt=Az+G(Y) \qquad (7)$$
$$Y=Cz$$

be the observer form; here G is some smooth vector function.

It is well known, see e.g.[5], that the observer synthesis especially easy in case when the system is given in the observer form. We show briefly how to obtain state estimates with help of observers.

Assume that the pair $(f;h,\rho)$ in some neighborhood of x^* is satisfied to some of sufficient conditions of solvability of the observer error linearization problem, see e.g. the paper [6]. In this case we can consider the pair (7) locally around zero instead of the pair (1) locally around x^*. Now in order to obtain a state estimate we can use the 1-parametric set of Luenberger-type observers

$$dw/dt=A_0w+G(y(t),a)$$

parametrized by the level set parameter a which is known for us.

5 On observability of a polynomial system restricted on a level set

Assume that the system (1) has distinguishable level sets of the first integral ρ. Then we can reduce observability analysis for the system (1) to observability analysis for the system (1) restricted on the level set corresponding to the output given. Note here that since the level set $\rho^{-1}(a)$ is an algebraic set we come to observability analysis of the system (1) restricted on the finite number of algebraic sets M_i being analytic manifolds of dimension $s<n$; here all $M_i\subset\rho^{-1}(a)$. It follows from the Lojasiewicz theorem on the stratification of semialgebraic sets, [8].

As a special case we have the situation when ρ is linear with respect to some variable, e. g. to x_n. We obtain a 1-parametric set of polynomial systems

$$dx_{(n-1)}/dt=F(x_{(n-1)};a), \qquad (8)$$
$$y=g(x_{(n-1)};a),$$

instead of (1) ; here $x_{(n-1)}=(x_1,\ldots,x_{n-1})\in\boldsymbol{R}^n$. So, since the parameter a is known we come to the observability problem for (8).

Now suppose that we are interested in the distinguishability condition of states taken from the same level set. We remark that the system (1) restricted on some level set Q is observable if and only if the mapping $(\rho,h;L_f^sh,s=1,2,\ldots)$ is injective. We can apply Theorem 1, [9], to the complexified system $dx^s/dt=f(x^s)$; $s=1,2$;

$(x^1,x^2)\in Q\times Q\cap\{(x^1,x^2)\mid h(x^1)=h(x^2)\}$.

Since the cited theorem is local we can apply its assertion for the dimension 2(n-1) and as a result we come to :

Proposition 2. Let $(f,h)\in V(\rho;n,d)\times\mathrm{OL}(n,l)$ and $\alpha=d^{r-1}l^r(l+d)^r$, with $r=2^{2n-4}$.

Then we have that the system (1) is observable on any level set Q of the first integral ρ if and only if $x\to(\rho(x),H_\alpha(x))$ is an injective mapping.

6 Concluding remark

One can generalize Theorem 1 for the case of multiinput-multioutput polynomial systems $dx/dt=f(x,u)$; $y=h(x)$ with many first integrals following the ideas of [2,7]. In this case by a first integral ρ we mean the polynomial vector-function $\rho=(\rho_1,\ldots,\rho_k)$ such that $L_f\rho_s(x)\equiv0$ for $s=1,\ldots,k$ and any admissible input *u*. Proposition 1 and 2 can be also generalized for the case of multioutput systems with many first integrals.

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