DC Motor Digital Control Using a Fractional Order Hold Device

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Abstract: - It is well known that the type of hold circuit used in digital control schemes critically influences the position of zeros of sampled-systems. Most digital control systems use a zero-order hold (ZOH). Nevertheless, they may be desirable alternatives to the ZOH reconstruction device in certain digital control problems. A FRactional Order Hold (FROH) circuit is applied here to a digitally controlled direct-drive DC motor (type QT6205C from Inland) described as a third order low-pass system. Significant improvements with respect to the ZOH case are provided when using a properly designed FROH signal reconstruction device.

Key-Words: - Digital control; fractional-order hold; sampled-data systems; stability, zeros.

1 Introduction

Unstable zeros limit the performance that can be achieved when controlling a system. Several techniques for control system design are based on the cancellation of process zeros. However, such methods can not be applied when the process has unstable zeros. Consequently, special attention is being paid to the study of the zeros of sampled-systems in recent years. This subject was studied by Amström et al. [1] for the case of Zero Order Hold (ZOH). Hagiwara et al. [2] carried out a comparative study demonstrating that a First Order Hold (FOH) provides no advantage over a ZOH as far as the stability of the zeros of the resulting sampled-data systems is concerned. Passino et al. [3] proposed the FRactional Order Hold (FROH) as an alternative to the ZOH. M. Ishitobi [4] analysed the stability properties of the limiting zeros -for sufficiently small or large sampling period- concluding that the FROH, properly adjusted, is not inferior to the ZOH in any case.

In this paper, a study of the stability properties of the discrete-time zeros of a direct-drive DC motor using a FROH signal reconstruction device is presented. In addition, two different approaches for the determination of the appropriate β will be proposed here. These approaches will allow us to obtain the FROH that provides the discretisation zeros as stable as possible -or with improved stability degree even when unstable- for a given plant and a given sampling period. The magnitude of the zeros obtained with such techniques is down to 60% smaller than the corresponding ZOH ones.

2 Theory

Suppose that the state space equation of an n^{th}-order time-invariant single-input single output controllable and observable low-pass system is expressed as

$$\begin{cases} \dfrac{dx}{dt} = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \qquad (1)$$
$$A \in R_{nxn}, B \in R_{nx1}, C \in R_{1xn}$$

where u(t) and y(t) are the input and the output scalars, and x(t) is the state vector. In order to design a digital control scheme, we are interested in the discrete-time system composed of the hold device, the continuous-time system and a sampler in series.

When the fractional-order hold (FROH) signal reconstruction device is used, the input is described by

$$u(t) = u(KT) + \beta\left[\frac{u(KT) - u((K-1)T)}{T}\right](t - KT) \qquad (2)$$
$$KT \le t < (K+1)T$$

where T is the sampling period and β is a device adjustable gain.

Then the sampled system is [3] [4]:

$$\begin{cases} \begin{bmatrix} x(KT+T) \\ x_1(KT+T) \end{bmatrix} = \begin{bmatrix} \Phi & \beta\Gamma^- \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} x(KT) \\ x_1(KT) \end{bmatrix} + \begin{bmatrix} \Gamma - \beta\Gamma^- \\ 1 \end{bmatrix} u(KT) \\ y(KT) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(KT) \\ x_1(KT) \end{bmatrix} \end{cases} \qquad (3)$$

where

$$\Phi = e^{AT}, \qquad \Gamma = \int_0^T e^{As} ds B, \qquad \Gamma^- = \int_0^T \left(\frac{s}{T}\right) e^{As} ds B$$

and 0 is an n^{th}-row vector with all zero elements. Thus the sampled transfer function is

$$G_\beta(z) = \frac{N_\beta}{D_\beta} = [C \quad 0]\left(zI - \begin{bmatrix} \Phi & \beta\Gamma^- \\ \mathbf{0} & 0 \end{bmatrix}\right)^{-1} \begin{bmatrix} \Gamma - \beta\Gamma^- \\ 1 \end{bmatrix} = \tag{4}$$
$$= \frac{\beta(z-1)\det\begin{bmatrix} zI - \Phi & -\Gamma^- \\ C & 0 \end{bmatrix} + z\det\begin{bmatrix} zI - \Phi & -\Gamma \\ C & 0 \end{bmatrix}}{z\det[zI - \Phi]}$$

where, according to (4), the location of the discrete-time poles does not depend on β.

Remark 1: Observe that the Zero Order Hold (ZOH) and First Order Hold (FOH) can be considered as particular cases of the FROH when β=0 and β=1, respectively. In doing so,

$$G_{T0}(z) = \frac{N_{T0}(z)}{D_{T0}(z)} = \frac{\det\begin{bmatrix} zI - \Phi & -\Gamma \\ C & 0 \end{bmatrix}}{z\det[zI - \Phi]} \tag{5}$$

and

$$G_{T1}(z) = \frac{N_{T1}(z)}{D_{T1}(z)} = \frac{(z-1)\det\begin{bmatrix} zI - \Phi & -\Gamma^- \\ C & 0 \end{bmatrix} + z\det\begin{bmatrix} zI - \Phi & -\Gamma \\ C & 0 \end{bmatrix}}{z\det[zI - \Phi]} \tag{6}$$

where $G_{T0}(z)$ and $G_{T1}(z)$ are the sampled transfer function with ZOH and FOH [3] respectively.

Remark 2: In addition, notice that a pole-zero cancellation in z=0 occurs when β=0 (ZOH). As a result of this cancellation, an order reduction effect will appear in the $G_{T0}(z)$ with respect to $G_ß(z)$ for any β≠0.

3 Zeros of sampled system

The zeros of a discrete-time system can be classified into two categories: those which correspond to the zeros of the continuous-time system, and those which do no have any continuous-time counterparts, introduced by the discretisation method. The zeros of the later category are called obviously discretisation zeros and are the kind of zeros that will appear when discretising any low-pass system.

The advantages of the FROH signal reconstruction device are verified through the design of a direct drive DC motor digital controller. A direct-drive motor is in fact just a very strong DC motor. Recently, DC motors that deliver a high torque on the motor axis have become available, due to the development of new magnetic materials. The high acceleration torque of these motors enables a direct coupling of the load to the motor axis, which makes a transmission superfluous. The disadvantages of the gear train, such as backlash and friction, can hence be avoided. Further, the use of permanent magnets instead of conventional field windings makes the direct-drive motor compact and its use attractive for robot applications.

An appropriate model for the direct-drive DC motor has the following differential equations [5]:

$$\begin{cases} u(t) = K_b\omega_p(t) + R_m i_a(t) + L_m \dfrac{di_a(t)}{dt} \\ M_m(t) = K_t i_a(t) \\ J_t \dfrac{d\omega_p(t)}{dt} + f\omega_p(t) = M_m(t) - M_\omega(t) \end{cases} \tag{7}$$

where R_m is the copper resistance, L_m is the motor inductance, K_t is the torque constant, K_b is the back-electromotive-force (EMF) constant, J_t is the total inertia (=J_l+J_m), J_l is the load inertia, J_m is the inertia of the motor axis, f is the viscous friction, M_m is the

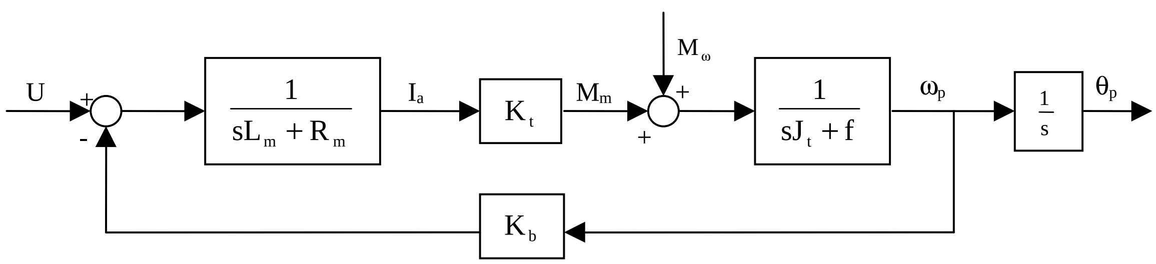

Figure 1. Model for the direct-drive DC motor.

motor torque, M_ω is the Coulomb friction, $\omega_p(t)$ is the angular velocity, $\theta_p(t)$ the angular position and u(t) is the input voltage. The Coulomb friction, which is caused by magnetic and mechanical hysteresis, results in a torque that has a magnitude M_ω that may be considered very small with respect to the motor torque M_m.

Taking the Laplace transform, a third order continuous-time low-pass transfer function is obtained

$$G(s)=\frac{\Theta_p(s)}{U(s)}=\frac{K_t}{(L_m J_t)s^3+(J_t R_m+fL_m)s^2+(R_m f+K_b K_t)s} \quad (8)$$

It can be seen from the figure 1 that the model contains an electrical transfer function $1/(sL_m+R_m)$ and a mechanical transfer $1/(sJ_t+f)$.

Using the values [5] $R_m=6.67\ \Omega$, $L_m=1.4x10^{-1}$ H, $K_t=3.26$ Nm/A, $K_b=3.26$ Vs/rad, $J_t=0.022$ Nms^2/rad and f =0.013 Nms/rad, the system is described as

$$G(s)=\frac{3.26}{(3.08x10^{-1})s^3+(0.14856)s^2+(10.71431)s}= \quad (9)$$
$$=\frac{1.0584x10^3}{s\left((s+24.1169)^2+53.8242\right)}$$

Our task is to design a digital controller that can be used to provide accurate angular position $\theta_p(t)$ of the direct-drive motor. Therefore, the first step consists of discretising the continuous plant. The system will be discretised by using a FROH device with $\beta \in [-1, 1]$ and a sampling period $T \in [0, 0.05]$ (seconds). In doing so, the discretisation zeros magnitudes can be determined by (4). The magnitude of the most unstable zero is presented in figure 2.

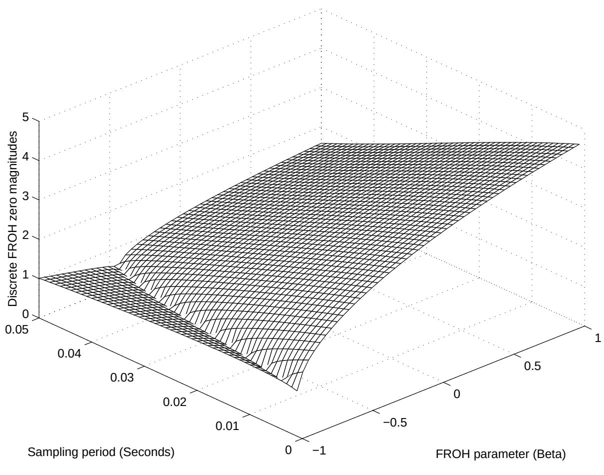

Figure 2. Magnitude of the most unstable zero.

In order to study the stability properties of this zero, a contour plot composed of lines of constant magnitude values obtained from the previous surface can be built. This contour is depicted in figure 3, where the stability region - commonly defined as $|z|\leq 1$ - has been shaded in grey. The improvement that can be reached by means of an appropriate choice of the parameter β, is clearly shown. The ZOH (β=0) and FOH (β=1) are not able to provide discrete-time zeros into this region for any sampling period $T \in [0, 0.054]$ seconds. However, by adjusting β to a suitable value in the FROH, an inverse stable discrete plant can be obtained for any T≥17 milisec.

The following properties can be derived from the previous study:

(i) For any given sampling period T, it is always possible to obtain FROH discretisation zeros that are more stable than the ZOH and FOH ones by means of an appropriate choice of negative values of the parameter β. This fact agrees with the theoretical properties of limiting FROH zeros -for sufficiently small (T→0) or large (T→∞) sampling periods - given by M. Ishitobi [4].

(ii) The contour lines limiting each region are very abrupt when negative values of β are used, that is, the magnitude changes very quickly with β. Therefore, the optimum value of β must be chosen as accurate as possible.

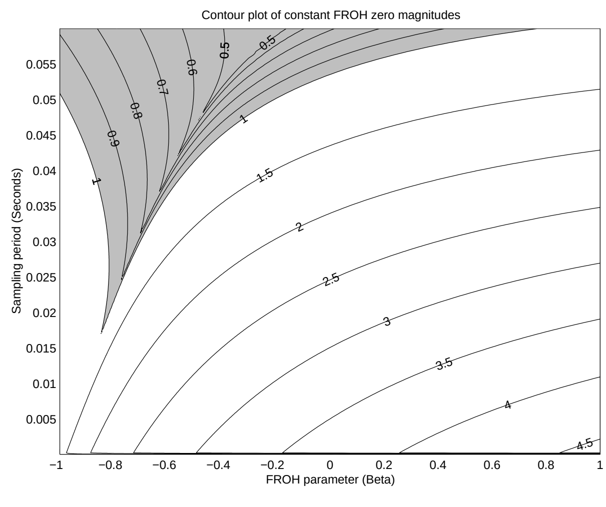

Figure 3. Contour plot of constant FROH zero magnitudes.

Two approaches are presented here in order to determine the optimum value of the parameter β, that is, the value that provides the FROH zeros as stable as possible for a particular continuos-time-plant and a particular sampling period T.

(a) The most general approach consists of obtaining the roots of Eqn (4) as a function of parameter β. Then, the value of β is adjusted in order to minimise the largest root magnitude by using approximated numerical recipes.

(b) The second approach consists of sketching the complementary generalised root loci. Then, by studying this root loci, the value of β corresponding to the most stable location of the discretisation zeros can be obtained.

Remark 3: The second approach is, fairly, the most advantageous. Nevertheless, sometimes this method is ill suited, depending on the continuous-time plant and, consequently, on the characteristics of the corresponding generalised root loci.

Now, the first method is applied to the case of the QT6205C direct-drive DC motor from Inland. For a given sampling period T=0.04 seconds, that is well-suited for our digital control system, the magnitude of he most unstable FROH zero as a function of β is plotted in figure 4. Then, the optimum value of β can be easily obtained by searching the minimum of that function (β_{opt}=-0.595 in our case). Once the β gain of the FROH reconstruction circuit is tuned to the value of β_{opt}, the most stable discretisation zeros are obtained for this particular continuous-time plant and a given sampling period.

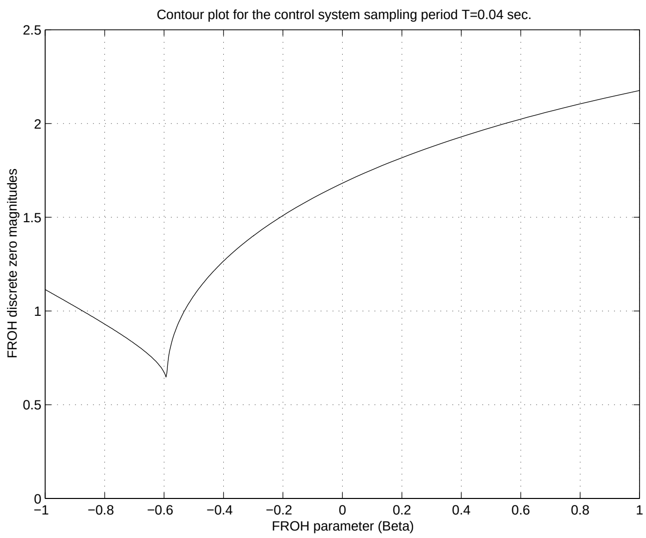

Figure 4. Contour plot of FROH zero magnitudes for the control system sampling period T = 0.04 seconds.

Notice that, in the case of the direct-drive DC motor digital controller under study, the FROH provides a most unstable discretisation zero magnitude ($|z_{opt}|$=0.6552) that is 61.04 % smaller than the magnitude corresponding to the ZOH (β=0) discretisation ($|z_{ZOH}|$=1.6820).

Finally, the generalised root-locus approach will be used to study the location of the discretisation FROH zeros on the complex plane.

According to equations (5,6)

$$N_{T0}(z) = \det\begin{bmatrix} zI-\Phi & -\Gamma \\ C & 0 \end{bmatrix} \tag{10}$$

and

$$N_{T1}(z) = (z-1)\det\begin{bmatrix} zI-\Phi & -\Gamma^- \\ C & 0 \end{bmatrix} + z\det\begin{bmatrix} zI-\Phi & -\Gamma \\ C & 0 \end{bmatrix} \tag{11}$$

where $N_{T0}(z)$ and $N_{T1}(z)$ are the numerators of the sampled transfer functions with zero order hold (ZOH) and first order hold (FOH) respectively. On the one hand, the roots of $N_\beta(z)$ are the zeros of $G_\beta(z)$. On the other hand, substituting Eqns (10, 11) into Eqn (4), the equation of the FROH discrete zeros $N_\beta(z)$=0 can be written as a generalised root loci with the parameter β as the generalised gain

$$1+\beta\left[\frac{N_{T1}(z)-zN_{T0}(z)}{zN_{T0}(z)}\right]=0 \tag{12}$$

where the generalised terminating points –generalised open-loop zeros- and starting points –generalised open-loop poles- are, respectively, the roots of

$$N_{T1}(z)-zN_{T0}(z) = (z-1)\det\begin{bmatrix} zI-\Phi & -\Gamma^- \\ C & 0 \end{bmatrix} \tag{13}$$

and

$$zN_{T0}(z) = z\det\begin{bmatrix} zI-\Phi & -\Gamma \\ C & 0 \end{bmatrix} \tag{14}$$

Applying Eqns (13,14) to the system described by Eqn (9), one obtains

$$1+\beta\left[\frac{(z-1)(z+3.3690)(z+0.2686)}{z(z+1.6820)(z+0.2093)}\right]=0 \tag{15}$$

Firstly, let's study the generalised root loci. The FROH discretisation zeros with β=[0, +∞) evolves rightwards on the negative real axis and leftwards on the positive real axes. Therefore, the FROH discretisation zeros with β>0 are always real and less stable than the ZOH ones (where β=0).

However, in the case of complementary generalised root loci β=[0, -∞), we have the same starting and terminating points, while the departure and arrival angles are opposite. Thus, there always exists a range of negative values of β for which the FROH zeros are located closer to the origin than the ZOH and FOH ones. This range goes from 0 to a value β_{max}, where β_{max} depends on the continuos-time plant parameters and the sampling period T. The complementary generalised root loci is depicted in Figure 5.

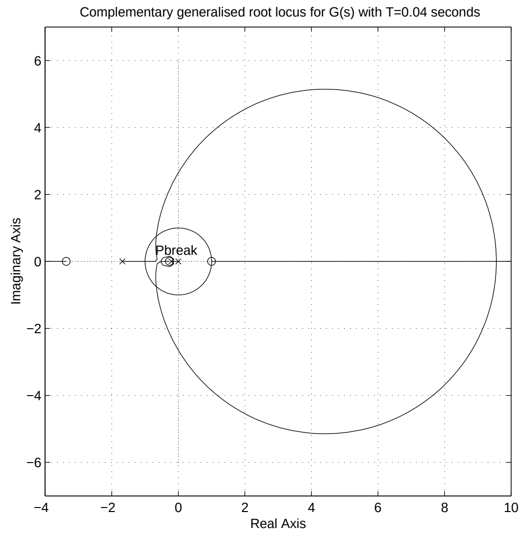

Figure 5. Complementary generalised root loci of G(s) for T=0.04 seconds.

We have four breakaway points located on the real axis at z_1=-0.6284, z_2=-0.5221, z_3=-0.1362 and z_4= 9.5543. The corresponding values of β are β_1=-0.5906, β_2=-0.5893, β_3=-0.1082 and β_4=-3.2989. In this case, as Fig.5 clearly shows, the optimum value of the parameter β corresponds to the breakaway point located in z_1=-0.6284 and marked in the figure as P_{break}. Therefore, $\beta_{opt}= \beta_1$ =-0.5906. Notice that this FROH provides a discretisation zero magnitude ($|z_{opt}|$=0.6284) that is 62.64 % smaller that the magnitude corresponding to the ZOH discretisation ($|z_{ZOH}|$=1.6820).

Besides, the FROH zeros are inside the stability region (unit circle in the figure), so that they can be cancelled by means of a digital controller. In this way, the performance of the controlled system is significantly improved [6] [7].

Remark 4: The small discrepancies denoted with respect to the results obtained using the first approach are due to the finite accuracy of the numerical recipes used to find the minimum of the function depicted in figure 4.

4 Conclusion

The location of the zeros introduced by the FROH signal reconstruction device on the complex plane has been studied for the case of a direct-drive DC motor controlled digitally. As a result of this study it has been shown that, using suitable values of the FROH parameter β, the sampled system zeros can be located inside the stability region in some cases when the ZOH (β=0) and FOH (β=1) fails to do so. Furthermore, for a given sampling period T, there exists a range of negative values of β for which the discretisation FROH zeros are up to 60% more stable that the ZOH and FOH ones. Finally, two approaches have been presented in order to determine the optimum value of the parameter β, that is, the value that provides the FROH zeros as stable as possible for a given continuos-time plant and sampling period. In the case of our DC motor, the magnitude of the zeros obtained with our technique is 62.64 % smaller than the corresponding ZOH ones. In this way, we can obtain an inverse stable sampled function in cases where ZOH and FOH generate unstable zeros. Therefore, this device suitably adjusted allow us to apply a wider range of digital control schemes and, consequently, to improve significantly the performance of the controlled system [6] [7].

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