The Propagation of Electromagnetic Waves in a Stratified Anisotropic Stochastic medium

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Abstract:- A direct investigation of the electromagnetic wave propagation in a stratified anisotropic medium, for which the parameters in the partial differential equations are piece-wise continuous functions of only one spatial variable, is presented. The formulation introduses a matrix describing the wave propagation. This approach enables us to decompose the wave field into up-going and down-going waves by implementing an eigenvector decomposition, much more simplified in comparison with the general case.
The matrix formulation method is used to calculate the electromagnetic response. Expanding properly the propagation matrix using Feynman's ordered exponential operators, we derive the electromagnetic response of a stratified anisotropic stochastic medium. The evaluation of the relevant stochastic integrals allows the determination of the stochastic properties of the electromagnetic response.

Key words:- electromagnetic, anisotropic, stochastic propagation.

1 Introduction

The interaction of electromagnetic waves with stratified media occurs in a wide variety of applications, from communication and optics down to the level of electronic components [1-3]. The basic problem refers the propagation of a plane electromagnetic wave, incident from the vacuum (or air) to stratified media, generally anisotropic. Their electrical properties (expressed by a tensor) vary in a stochastic manner at the vertical direction, defined by z coordinate axis but not depend on x or y.

The electromagnetic transmission into layered stochastic isotropic media has been recently studied by Vallianatos [4], who employed a 2x2 propagation matrix. Note that this propagation matrix, for a uniform layer could be explicitly found in terms of the solution of an algebraic problem.

The present work generalizes the latter approach. It introduces a 4x4 matrix appropriate to describe the electromagnetic wave propagation in a stratified anisotropic structure. In the frame of this approach the wavefield is decomposed into up-going and down-going waves using an eigenvector decomposition, much more simplified compared to the general case.

Consequently the matrix formulation method is used to calculate the electromagnetic response. The propagation matrix, is properly expanding using Feynman's ordered exponential operators, and the electromagnetic response of a stratified anisotropic stochastic medium is derived. The evaluation of the relevant stochastic integrals results to the determination of the stochastic properties of the electromagnetic response.

2 The Dominant Equations

For the general approach, a set of Cartesian axes Oxyz is adopted, with the z-axis pointing downwards in the vertical direction. The structure is bounded in depth by a half-space with complex conductivity:

$$\boldsymbol{\sigma}_{\mathrm{b}} = \begin{bmatrix} \sigma_{b1} & 0 \\ 0 & \sigma_{b2} \end{bmatrix} = (\sigma_{bij}) - i\omega(\varepsilon_{bij})$$

The basic Maxwell equations prevailing the problem are expressed by:

$$\nabla \times \mathbf{E} = i\,\omega\mu\,\mathbf{H} \quad \text{and}$$
$$\nabla \times \mathbf{H} = \boldsymbol{\sigma}\,\mathbf{E} \qquad (1)$$
where ω is the angular frequency of the electromagnetic field, μ the magnetic permeability, which is supposed to be that of the vacuum μ_o and $\boldsymbol{\sigma}$ the complex conductivity tensor.

Introducing a vector

$$\boldsymbol{\Psi} = (Ex, Hy, Ey, -Hx)^T$$

(similar of that used by Berreman [5] and Abramovici [6] the set of equations (1) can be written in a matrix form

$$\frac{\partial \Psi}{\partial z} = \mathbf{A}\,\boldsymbol{\Psi} \text{ where}$$

$$\mathbf{A} = \begin{bmatrix} 0 & i\omega\mu & 0 & 0 \\ -\sigma_{11} & 0 & -\sigma_{12} & 0 \\ 0 & 0 & 0 & i\omega\mu \\ -\sigma_{21} & 0 & -\sigma_{22} & 0 \end{bmatrix} \qquad (2)$$

Introducing both the electric and magnetic field components in the propagation equation, we have the advantage expressing the boundary conditions at the interface, without explicitly involving the derivatives of the electric field at the surface itself. The general solution of equation (2) can be formally expressed using the ordered exponential operator

$$\mathbf{F}(z,0) = EXP(\int_0^z \mathbf{A}(z)\mathrm{d}z) \qquad (3)$$

originally introduced by Feynman [7]. Then $\boldsymbol{\Psi}(z)$ can be written in the form:

$$\boldsymbol{\Psi}(z) = \mathbf{F}(z,0)\boldsymbol{\Psi}(0) \qquad (4),$$

where $\mathbf{F}(z,0)$ is the so-called propagator matrix, used by Gilbert and Backus [8] in the study of elastic wave propagation.

In order to calculate the ordered exponential operator we follow the so-called Magnus approximation [9]. According to the Magnus approach: $\mathbf{F}(z,0)=\exp[\boldsymbol{\Omega}(z,0)]$ where the exponent is an infinite series, the first terms of which are :

$$\boldsymbol{\Omega} = \int_0^z \mathbf{A}(z)dz + \frac{1}{2}\int_0^z [\mathbf{A}(z_1), \int_0^{z1} \mathbf{A}(\mathbf{z_2})\mathrm{d}z_2]dz_1 + \qquad (5)$$

$$+\frac{1}{4}\int_0^z [\mathbf{A}(z_1), \int_0^{z_1} [\mathbf{A}(z_2), \int_0^{z_2} \mathbf{A}(z_3)dz_3]dz_2] + \dots\dots\dots$$

where $[\mathbf{A},\mathbf{B}|=\mathbf{AB}-\mathbf{BA}$ the commutator of the operators $\mathbf{A}$ and $\mathbf{B}$ [10]. We point out that expression (5) is the continuous analog of the Baker-Hausdorff formula[11,12].

Expression (4) can be used for the determination of the electromagnetic response. If the elements of the matrix $\mathbf{F}(L,0)$ are known and by considering that on that on the surface of the half-space basement with tensor complex conductivity $\boldsymbol{\sigma}_b$ hold

$H_y(L)=(k_{b1}/\omega\mu)E_x(L)$ and

$H_x(L)=-(k_{b2}/\omega\mu)E_y(L)$

where $k^2_{bi}=i\omega\mu\sigma_{bi}$, a linear system of algebraic equations is obtained. The solution of thesystem provides the elements of the electromagnetic response tensor through the relations:

$$Zxx = \qquad (6)$$
$$\frac{(F_{14}-\xi_1F_{24})(F_{33}-\xi_2F_{43})-(F_{34}-\xi_2F_{24})(F_{13}-\xi_1F_{23})}{D}$$

$$Zxy =$$
$$\frac{(\xi_1F_{22}-F_{12})(F_{33}-\xi_2F_{43})-(\xi_2F_{42}-F_{32})(F_{13}-\xi_1F_{23})}{D}$$

$$Zyx =$$
$$\frac{(F_{11}-\xi_1F_{21})(F_{34}-\xi_2F_{24})-(F_{31}-\xi_2F_{41})(F_{14}-\xi_1F_{24})}{D}$$

$$Zyy =$$
$$\frac{(\xi_2F_{42}-F_{23})(F_{11}-\xi_1F_{21})-(\xi_1F_{22}-F_{12})(F_{31}-\xi_2F_{41})}{D}$$

where

$D=(F_{11}-\xi_1F_{21})(F_{33}-\xi_2F_{43})-(F_{31}-\xi_2F_{41})(F_{13}-\xi_1F_{23})$

and $\xi_i=\omega\mu/k_{bi}$.

In the case of a structure which is described by a complex conductivity of the form $\boldsymbol{\sigma}=\sigma(z)\mathbf{I}_2$ where $\mathbf{I}_2=(\delta_{ij})$ is the 2x2 identity matrix, overburden a homogeneous isotropic half-space with complex conductivity σ_b the aforementioned expressions simplified to:

$$Zxy(z=0) = \left.\frac{Ex}{Hy}\right|_{z=0} \qquad (7)$$

$$= \frac{F_{22}-F_{12}(k_b/\omega\mu)}{-F_{21}-F_{11}(k_b/\omega\mu)} = -Zyx(z=0)$$

The analysis is proceed to the calculation of the propagation matrix $\mathbf{F}$ in the case of a simple isotropic layered structure, overlying a half-space basement with conductivity σ_b. In each layer of thickness h_m, in which $\mathbf{A}(z)$, can be considered constant equation (2) can be exactly solved [13].

$\boldsymbol{\Psi}(L_m)=\exp(\mathbf{A}h_m)\boldsymbol{\Psi}(L_{m-1})=\mathbf{P}_m(L_{m-1},h_m)\boldsymbol{\Psi}(L_{m-1})$ (8)

where,

$\mathbf{P}_m(L_{m-1},h_m)=\exp(\mathbf{A}.h_m)$ and $\mathbf{L}_\kappa=h_1+h_2+\dots\dots\; h_k$ .Using the Cayley-Hamilton (CH) theorem [14] and expression (5), an analytic expression for the propagation matrix could be algebricaily calculated. For the case of isotropic layered structure, where $\boldsymbol{\sigma}_m = \sigma_m \mathbf{I}_2$ the propagation matrix $\mathbf{P}_m$ has the form :

$$\mathbf{P}_m(L_{m-1}, h_m) = \qquad (9)$$

$$= \begin{bmatrix} \mathbf{Q}(L_{m-1}, h_m) & \mathbf{O_2} \\ \mathbf{O_2} & \mathbf{Q}(L_{m-1}, h_m) \end{bmatrix}$$

where

$$\mathbf{Q}(L_{m-1}, h_m) =$$

$$= \begin{bmatrix} \cosh(ik_m h_m) & (\omega\mu / k_m)\sinh(ik_m h_m) \\ (i\sigma_m / k_m)\sinh(ik_m h_m) & \cosh(ik_m h_m) \end{bmatrix}$$

and $k^2_m = i.\omega\mu\, \sigma_m$. O_2 is the 2x2 null matrix. The latter expression can be used for exact determination of the electromagnetic response in the case of a stratified structure. In fact, if h_1, h_2, ..h_{n-1}, the thicknesses of the n-1 layers, respectively, overburden of the halfspace, we obtain :

$$\boldsymbol{\Psi}(L) = \mathbf{F}(L,0)\boldsymbol{\Psi}(0) = (\mathbf{P}n\text{-}1.\mathbf{P}n\text{-}2. \ldots \mathbf{P}_1)\boldsymbol{\Psi}(0) \quad (10)$$

where

$$L = \sum_{m=1}^{n-1} h_m$$

It is already mentioned that using the afore-mentioned propagation matrix method, a 4x1 generalized field vector $\boldsymbol{\Psi}$ is defined with a 4x4 propagation matrix **F**, which describes the propagation of the generalized electromagnetic field vector. In a particular frequency of the electromagnetic field the propagation matrix depends, as expected, on the subsurface complex conductivity structure.

An alternative presentation of the propagation matrix is assessed assuming that the eigenvalues of the matrix $\boldsymbol{\Omega}$ are distinct. Therefore, applying the CH theorem, a matrix **U** is introduced. The columns of **U** are the eigenvectors of the matrix $\boldsymbol{\Omega}$, that has the property $\boldsymbol{\Omega} = \mathbf{U}\, \boldsymbol{\Lambda}\, \mathbf{U}^{-1}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix, with diagonal elements the eigenvalues λ_i of $\boldsymbol{\Omega}$. Thus, the propagation matrix **F** can be found, through the eigenvectors and eigenvalues of Ω as:

$$\mathbf{F} = \exp(\boldsymbol{\Omega} h) = \mathbf{U}\exp(\boldsymbol{\Lambda} h)\mathbf{U}^{-1} = \mathbf{U}\mathbf{K}\mathbf{U}^{-1} \qquad (11)$$

where **K** is also a diagonal matrix with elements $K_{ij} = \delta_{ij} \exp(\lambda_i\, h)$ determined by the four eigen-values λ_i of $\boldsymbol{\Omega}$. We define now a new generalized electromagnetic field vector $\boldsymbol{\Phi}(z) = \mathbf{U}^{-1}\boldsymbol{\Psi}(z)$. The new field vector decomposes the wave equation (2) into four independent equations

$$\frac{\partial \Phi}{\partial z} = \boldsymbol{\Lambda}\boldsymbol{\Phi} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \Phi$$

The four independent equations have the elementary solution of the form :

$\Phi_i(z+h) = \exp(\lambda_i h)\boldsymbol{\Phi}_i(z)$ or equivalently $\boldsymbol{\Phi}(z+h) = \mathbf{K}\boldsymbol{\Phi}(z)$. From the latter expressions it is obvious that, since the new field $\boldsymbol{\Phi}$ has been determined, the generalized electromagnetic field vector $\boldsymbol{\Psi}$ is given by:

$$\boldsymbol{\Psi}(z+h) = \mathbf{U}\boldsymbol{\Psi}(z+h) = \mathbf{U}\mathbf{K}\mathbf{U}^{-1}\boldsymbol{\Psi}(z)$$

which is a combination of equations (8) and (11).

3 The electromagnetic response of a stratified stochastic medium

In the case of a stochastic medium the complex conductivity tensor $\boldsymbol{\sigma}$ is supposed to vary stochastically in space with given statistical properties (e.g. as a random variable). The stochastic medium study deals with the statistical quantities of the medium and the relevant statistical quantities of the wavefield.

To simplify the algebra we consider the particular case of a scalar medium conductivity which depends on the z- coordinate only (stratified medium).

Furthermore, in the previous section we showed that if the characteristic propagation matrix **F**(z,0) of the medium is known, the electromagnetic response can be easily obtained.

Thus, we proceed to the theoretical study of the electromagnetic field in the case of a stratified stochastic structure, composed of a layer with thickness L, characterized by a stochastic variation of its complex conductivity, which overlies a half space basement having a constant complex conductivity and the calculation of the electromagnetic response at the surface of the structure.

Let us calculate the propagation matrix **F**(L,0) for a layer of thickness L with conductivity $\sigma(z) = \sigma_o + \xi(z)$, where σ_o represents the average conductivity of the layer and $\xi(z)$ is a random uncorrelated zero-mean function of the z-coordinate. We attempt to approximate equation (5). Writing explicitly the first commutators of the expansion of the ordered exponential (5) we keep the first two terms. Therefore in the approximation order $(kL)^2$, i.e when the penetration depth of the electromagnetic field is greater than L, we obtain

$$\mathbf{F}(L,0) = \tag{12}$$

$$= \exp\begin{bmatrix} (i\omega\mu/2)(S_2 - S_1) & i\omega\mu L \\ -(\sigma_o L + S_1) & -(i\omega\mu/2)(S_2 - S_1) \end{bmatrix}$$

where

$$S_1 = \int_0^L \xi(z)dz \quad S_2 = \int_0^L z\xi(z)dz$$

$$S_3 = \int_0^L dz(\int_0^z \xi(z_1)dz_1$$

Using the CH theorem the elements F_{ij} of the matrix exponential of equation .(12) can be easily expressed as :

$$F_{11}(L,0) = \cosh(ik_o L\alpha) +$$
$$+ \frac{\omega\mu}{2k_0 L\alpha}(S_2 - S_3)\sinh(ik_o L\alpha)$$

(13)

$$F_{12}(L,0) = \frac{\omega\mu}{k_0\alpha}\sinh(ik_o L\alpha)$$

$$F_{21}(L,0) = \frac{-(\sigma_o L + S_1)}{ik_0 L\alpha}\sinh(ik_o L\alpha)$$

$$F_{22}(L,0) = \cosh(ik_o L\alpha) -$$
$$- \frac{\omega\mu}{2k_0 L\alpha}(S_2 - S_3)\sinh(ik_o L\alpha)$$

where $k^2_o = i.\omega\mu\sigma_O$ and

$$\alpha = \left[1 + \frac{S_1}{\sigma_o L} - \frac{i\omega\mu}{4\sigma_o L^2}(S_2 - S_1)^2\right]^{1/2}$$

To demonstrate the calculation of the electro-magnetic response, we assume a structure consisting of a layer with thickness L having stochastic variation of the complex conductivity over a half-space with conductivity σ_b In this configuration the electromagnetic response can be calculated by using eqs.(7) and (13) as :

$$Zxy(z=0) =$$

$$= \frac{\cosh(ik_o L\alpha) - \left[\frac{k_b}{\alpha_0} + \frac{\omega\mu(S_2 - S_3)}{2k_o L\alpha}\right]\sinh(ik_o L\alpha)}{\frac{k_b}{\omega\mu}\cosh(ik_o L\alpha) + \left[\frac{\sigma_o L + S_1}{ik_o L\alpha_0} + \frac{k_b(S_2 - S_3)}{2k_o L\alpha}\right]\sinh(ik_o L\alpha)}$$

which in the absence of the stochastic perturbation (i.e $\xi(z)=0$) exactly coincides with the well-known results for a homogeneous layer of thickness L overlying a half-space with conductivity σ_b [15]. Note, that keeping higher order corrections in equation (13) the resulting estimations for Zxy(z=0) do not present any substantial difference with the values estimating using equation.(14).

The propagator matrix **F**(L,0) and consequently the electromagnetic response can found when the stochastic integrals $S_1(L)$, $S_2(L)$ and $S_3(L)$ are known. For a given configuration of the stochastic fluctuation $\xi(z)$, these integrals can be exactly estimated by numerical integration. However, in some particular cases, it is possible to derive useful information about the statistical properties of the electromagnetic response of a structure through an analytical evaluation of the relevant stochastic integrals. For this purpose we assume in the following, that the stochastic perturbation $\xi(z)$ of the conductivity of the upper layer has a Gaussian probability distribution with zero average and ε^2 standard deviation. Using the basic techniques of Statonovich stochastic calculus [16,17,18] it is not difficult to demonstrate that:

$$S_1 = \int_0^L \xi(z)dz = LY_1$$

$$S_2 = \int_0^L z\xi(z)dz = \frac{L^2Y_1}{2} - \frac{L^2Y_2}{2\sqrt{3}}$$

$$S_3 = \int_0^L dz(\int_0^z \xi(z_1)dz_1 = \frac{L^2Y_1}{2} + \frac{L^2Y_2}{2\sqrt{3}}$$

where Y_1 and Y_2 are two independent Gaussian stochastic variables with zero average and standard deviation ε^2. We emphasize that the latter equations should be intended in the sense that, given a series of independent realizations of $\xi(z)$ the statistical distribution of the right hand side and left hand side of the equations are the same. Substitution of the corresponding terms in equation (13) allows us to write for the probability distribution of Zxy(z=0)

$$\Pr ob[Zxy(z=0)] =$$

$$= \Pr ob\left[\frac{\cosh(ik_o L\alpha) - \left[\dfrac{k_b}{\alpha_0} + \dfrac{\omega\mu L Y_2)}{2k_o\alpha\sqrt{3}}\right]\sinh(ik_o L\alpha)}{\dfrac{k_b}{\omega\mu}\cosh(ik_o L\alpha) + \left[\dfrac{\sigma_o + Y_1}{ik_o\alpha_0} + \dfrac{k_b L Y_2}{2\sqrt{3}k_o\alpha}\right]\sinh(ik_o L\alpha)}\right]$$

where

$$\alpha = \left[1 + \frac{Y_1}{\sigma_o} - \frac{(k_0 L)^2}{12}\frac{Y_2}{\sigma_o}\right]^{1/2}$$

From the application of a Monte Carlo simulation in order to estimate the probability distribution of the module of Zxy(z=0) for different frequencies and degrees of perturbation (i.e., different ε^2) [4] is drawn the conclusion that as the frequency decreases, the width of the probability distribution bell is shortened and the curve becomes less rough. This indicates that the influence of the stochastic complex conductivity term is stronger as the frequency of the electromagnetic wave increases. From physical point of view it is inferred that as the frequency decreases, the influence of the upper stratified stochastic structure is getting weaker and the electromagnetic response is determined mainly from the field distribution in the underlying medium.

4 Concluding remarks

In the present paper, we define a 4x4 propagation matrix. using a generalized electromagnetic field vector **Ψ.** Using the element of this matrix, the electromagnetic response tensor for a stratified anisotropic structure, is calculated. Also an alternative presentation of the propagation matrix is given based on a diagonalization of the characteristic equation. Introducing a new 4x1 field vector, the wave equation is decomposed into four independent wave equations, having solutions in an elementary matrix form. An advantage of the proposed calculation procedure, from the theoretical point of view, is that uses only matrix multiplications, instead of a differential equation system.

Furthermore, using the aforementioned field vector **Ψ**, the problem of the propagation of electromagnetic field in a stratified stochastic medium is studied. It is conducted for a structure in which an upper layer characterized by stochastic variance in its complex conductivity exists. Our approach is based on the expansion to the approximation order $(kL)^2$, of the electromagnetic propagation matrix according to the Magnus technique. In the frame of this approach it is possible to evaluate the statistical properties of the electromagnetic response Zxy(z=0) of our layered configuration. We point out that, the extension of this method to the calculation of higher order corrections to the present results, does not present any substantial differences.

References

[1] H. Bertoni and L. Felsen, *Directions in Electro magnetic Wave Modeling*, Plenum, 1991

[2] R. King, M. Owens and T.T.Wu, *Lateral Electromagnetic Waves*, Springer-Verlag, 1992.

[3] M. Born and E. Wolf, Principles of Optics, Pergamon Press, 1970.

[4] F. Vallianatos , *Magnetotelluric response of a random layered earth,* Geophys. J.. Int., Vol.125, 1996, pp.557-583.

[5] D. Berreman, Optics in stratified and aniso-tropic media, *J. Opt. Soc. Am.,Vol.* 62, 1972, pp.502-510.

[6] F. Abramovici, The forward magnetotelluric problem for an inhomogeneous and anisotropic structure, *Geophys.*, Vol. 39, 1974, pp. 56-68..

[7] R. Feynman, An operator calculus having applications in quantum electrodynamics, *Phys. Rev.*, Vol. 84 No. 1, 1951, pp. 108-128.

[8] F. Gilbert and G. Backus, Propagation matrices in elastic waves and vibration problems, *Geophys.,Vol.* 31, 1966, pp. 326-332.

[9] W. Magnus, On the exponential solution of differential equations for a linear operator, *Commun. Pure Appl. Maths.*, Vol. 7, 1954, pp. 649-673.

[10] L. I. Schiff, Quantum Mechanics, McGraw Hill, 1982.

[11] H. F. Baker, On the integration of linear differential equations, *Proceedings of the London Mathematical Society*,Vol. 2, 1904, pp. 293-296.

[12] K. O. Fredrchs, Mathematical aspects of the quantum theory of fields, Part V., *Commun. Pure Appl. Maths.*,Vol. 6, 1953, pp. 1-72.

[13] R. Bronson, *Modern introductory Differential Equations*, McGraw-Hill, 1973.

[14] P. Kahn, *Mathematical methods for scientists and engineers. Linear and non-linear systems*, John Wiley, 1990.

[15] J. Wait, Electromagnetic Waves in Stratified Media, Pergamon Press, 1970.

[16] C. W. Gardiner, *Handbook of stochastic methods*, Springer, 1985.

[17] B. Oksendal, *Stochastic differential equations*, Springer - Verlang, 1992.

[18] S. K. Srinivasan. and K. M. Merata, *Stochastic processes*, McGraw- Hill, 1988.