Some Comments on Bifurcation for Driftless Systems: Existence of Solutions and Uniqueness

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Abstract

In this paper we discuss the existence and uniqueness of solution in the neighborhood of an ideal one for driftless system.

This problem is very important, since generally the model used for the control design was deduced from some approximation. The objective of this paper is to show that some particular inputs and quadratic terms may render the control problem ill posed, in the sense that the real solution may be not unique or does not exist in the neighborhood of the expected one.

Keywords: System theory, Bifurcation, Nonlinear dynamics, Solution analysis.

1 Introduction

In control theory, the type of bifurcation studies is generally with respect to stability [1] or controllability [7]. Here, we propose a work preliminary to these problems, we discuss the existence and uniqueness of solution in the neighborhood of an ideal one and this for driftless system. Generally the model used for the control design was deduced from some approximation. So, some forgotten terms give, for particular inputs, dynamical solution "very fare" to the expected one. Roughly speaking, we design our control law on the base of an approximated model which gives an expected solution and unfortunately, for some particular inputs and quadratic terms, the real solution may be not unique or does not exist in the neighborhood of the expected one. Obviously, this preliminary work is very important in control theory, because stability, controllability, an analyze based on the expected solution may be totally irrelevant. Thus, our approach is based on bifurcation theory [6, 10, 12] and so in the neighborhood of the dominant order terms [1] "linear" solution of the problem [2], we discuss the number of solutions to our nonlinear control problem.

In section two, we state the problem of the existence and number of solutions to the integration for driftless systems. In section three we reformulate the problem abstractedly. In the next section, we analyze the existence and the uniqueness of the solution. We end this section with our main result which highlights the fact that we may have no solution, one solution or two solutions, in the neighborhood of the expected ("linear") solution, in function of the system structure and the input choice. Thus, we exhibit a bifurcation phenomenon from a nontrivial solution associated with the "linear" dynamics. We conclude by some comments and perspectives related to the necessity of such a preliminary discussion before the analysis of the analyze system stability, controllability, etc.

2 Problem statement

Let us consider the following driftless system form

$$\dot{x} = g_1(x)\bar{u}_1 + g_2(x)u_2 \tag{1}$$

where $x \in \mathbb{R}^n$, $\bar{u}_1$, $u_2 \in \mathbb{R}$ and $g_1(x)$, $g_2(x)$ are C^k vector fields for some sufficiently large k and $t \in I = [0, T]$, where $T > 0$ is fixed.

We suppose, that the following assumption is always verified:

Hypothesis (H.1)

$$Rank\{span\{g_1, g_2, ad_{g_1}g_2, \ldots, ad_{g_1}^{n-2}g_2\}\} = n$$

[1] Here, we use the extension of Poincaré forms to controlled dynamics [8, 7].

[2] linear with respect only to a subpart of the state and input

The assumption (H.1), implies that $Rank(g_1(0), g_2(0)) = 2$. Consequently there exist (see [2]) a diffeomorphism and a prefeedback on $\bar{u}_1 = \beta(x, u_1)$, $\beta \in C^k$ with respect to x and u_1. Moreover, as $\frac{\partial \beta}{\partial u_1} \neq 0$, the system (1) is equivalent to

$$\dot{z}_1 = u_1 \tag{2a}$$
$$\dot{\tilde{z}} = A(z_1)\tilde{z}u_1 + O(\tilde{z})^2 u_1 + B(z_1)u_2 + O(\tilde{z})u_2 \tag{2b}$$

where $\tilde{z} = (z_2, ..., z_n)$, $A(z_1), B(z_1) \in \mathbb{R}^{n-1}$, $A(z_1)$, $B(z_1)$ are continuous on I, $O(\tilde{z})$ is at least linear on $\tilde{z}$ and $O(\tilde{z})^2$ is at least quadratic on $\tilde{z}$.

Moreover, as our purpose here is not to study the stabilizability of (2) (for this, see for example [5]) we assume that the control law takes the following form :

C-1 $u_1 = f(z_1, t)$, f is continuous and $z_1(0)$ is fixed.

Remark: As u_1 is continuous with respect to both variables and moreover z_1 is continuous with respect to t, u_1 is obviously continuous-time function and this assumption is enough for our purpose.

C-2 $u_2 = h(z_1, \tilde{z})$ is C^2 in $\tilde{z}$ and $h(z_1, 0) = 0$

Remark: Control constraints imply that there exist two matrices $F(z_1)$ and $G(z_1)$ of dimension $n \times n$, such that:

$$u_2 = F(z_1)\tilde{z} + \tilde{z}^T G(z_1)\tilde{z} + O(\tilde{z})^3 \tag{3}$$

Consequently, we rewrite (2) as follows :

$$\left\{ \begin{array}{lll} \dot{z}_1 & = & f(z_1, t) \triangleq u_1 \\ \dot{\tilde{z}} & = & A(z_1)\tilde{z}u_1 + O(\tilde{z})^2 u_1 \\ & + & (B(z_1) + O(\tilde{z}))(F(z_1)\tilde{z} + \tilde{z}^T G(z_1)\tilde{z}) \end{array} \right.$$

Moreover, by convention we have $\tilde{z}(0) \triangleq \tilde{z}_0$ and $\tilde{z}_0$ is in the neighborhood of zero in $\mathbb{R}^{n-1}$.

In the paper, the main key point is to characterize the influences of the quadratic terms in $\tilde{z}$ (and this in function of u_1) with respect to the existence and the uniqueness of the solution for system (2) . Thus, we note by γ the C^2 function with respect to $\tilde{z}$ defined from $\mathbb{R}^{n-1} \times \mathbb{R} \times I$ to $\mathbb{R}^{n-1}$ which is equal to

$$\begin{array}{lll} \gamma(\tilde{z}, u_1, t) & = & O(\tilde{z})^2 u_1 + B(z_1)\tilde{z}^T G(z_1)\tilde{z} \\ & + & O(\tilde{z})(F(z_1)\tilde{z} + \tilde{z}^T G(z_1)\tilde{z}) \end{array}$$

Thanks to γ, hereafter we will formulated abstractedly the problem, this greatly simplifies our discussions.

3 Abstractedly problem formulation

First, it is important to note that we discuss and particularize with respect to system (2) the result obtained in [3] in a more general context.

Let $Y = C(I, \mathbb{R}^{n-1})$ be the set of continuous functions defined on time interval I, having values in a subspace $\mathbb{R}^{n-1}$,taking the uniform convergence norm given by:

$$y \in Y : ||y||_\infty \triangleq \sup_{t \in I} ||y(t)||_0 \text{ where } ||.||_0 \text{ is a norm on } \mathbb{R}^{n-1}$$

$(Y, ||.||_\infty)$ is a Banach space ([6]).

Moreover, taking $X = C^1(I, \mathbb{R}^{n-1})$ the set of C^1 functions defined on the time interval I having values in subspace $\mathbb{R}^{n-1}$, we set the norm for X as follows:

$$x \in X \quad ||x|| = ||x||_\infty + ||\dot{x}||_\infty$$

and thus $(X, \mathbb{R}^{n-1})$ is also a Banach space .

Let $U \subset C(I, \mathbb{R})$ be the set of continuous controls on u_1, we define the linear operator L by:

$$\begin{array}{llll} L: & X \times U & \rightarrow & Y \\ & (\tilde{z}, u_1) & \rightarrow & L_{A(z_1),B(z_1),F(z_1)}(\tilde{z}, u_1)(t) = \tilde{z} - \phi(t)\tilde{z}(0) \end{array}$$

where $\phi(t)$ is the fundamental matrix (flow matrix) associated with the linear subsystem [3] :

$$\left\{ \begin{array}{lll} \dot{\tilde{z}} & = & (A(z_1)u_1 + B(z_1)F(z_1))\tilde{z} \\ \tilde{z}(0) & = & \tilde{z}_0 \end{array} \right. \tag{4}$$

and we define the nonlinear operator N by:

$$\begin{array}{llll} N: & X \times U & \rightarrow & Y(\tilde{(z)}, u_1) \\ & (\tilde{z}, u_1) & \rightarrow & N_{G(z_1),B(z_1),F(z_1)}(t) \\ & & & = \int_0^t \phi(t)\phi^{-1}(s)\gamma(\tilde{z}(s), u_1(s), s)ds \end{array}$$

Remarks:

- In the definition of N, $\phi^{-1}(s)$ always exists and this is the flow solution at time $-s$.
- For the sake of simplicity, we note L in stead of $L_{A(z_1),B(z_1),F(z_1)}$ and N instead of $N_{G(z_1),B(z_1),F(z_1)}$. $\triangle$

From this, we immediately obtain :

Proposition 3.1 *Problem (2) is equivalent to:*

$$L(\tilde{z}, u_1) = N(\tilde{z}, u_1) \text{ where } \tilde{z} \in X \text{ and } u_1 \in U \tag{5}$$

in the sense that each solution of (2) is solution of problem (5) and inversely.

The proof follows immediately expressing in (5) the two operators thanks to the Definition. Thus deriving (5) we obtain (2) and we have the same solution on X, for all fixed $u_1 \in U$.

[3] In the Poincaré form the order 1 with respect to $\tilde{z}$ and z_1, u_1 considered as order zero term.

4 The existence and uniqueness of solutions

4.1 The case of $\tilde{z}_0 = 0$

As L is invertible on Y for u_1 being fixed [4] , thus equation (5) is equivalent to:

$$\tilde{z} - N(\tilde{z}, u_1) = 0$$

We note :

$$\begin{array}{rccl} M: & X \times U & \to & X \times U \\ & (\tilde{z}, u_1) & \to & M(\tilde{z}, u_1) = \tilde{z} - N(\tilde{z}, u_1) \end{array}$$

M verifies :

$$\begin{array}{lll} i) & M(\tilde{z}, u_1) & = & 0 \\ ii) & M(0,0) & = & 0 \\ iii) & D_{\tilde{z}} M(.,0)|_0 & = & I_d|_X, \end{array} \tag{6}$$

where $D_{\tilde{z}}$ is the Frechet partial derivative of M in relation with to $\tilde{z}$.

The theorem of implicit functions is applied and consequently equation (6) admits a **unique** solution $\tilde{z}^* = \tilde{z}^*(u_1)$ such that $z^*(0) = 0$ and $\tilde{z}^*(u_1) - N(\tilde{z}^*(u_1), u_1) = 0$ for each u in a neighborhood of 0 in R where $\tilde{z}^*$ is continuous with respect to u_1.

We resume this result in the following theorem:

Theorem 4.1 *For $\tilde{z}_0 = 0$, the problem (2) admits a* **unique** *solution $\tilde{z}^*$, locally in the neighborhood of 0 in X, with $\tilde{z}^*(0) = 0$, $\tilde{z}^*$ is continuous in u_1, u_1 being in a neighborhood of 0 in U.*

Remark: This case corresponds, for example, to a case which has only its velocity different from zero. This is not the most important case and obviously, in this solution the bifurcation does not appear.

Consequently, we focus our attention now to the other case (i.e. $z_0 \neq 0$).

4.2 The case of $\tilde{z}_0 \neq 0$: Bifurcation Analysis

When $\tilde{z}_0 \neq 0$, the linear problem (4) obviously admits a nontrivial solution noted $\tilde{Z}_0$, and such that $\tilde{Z}_0(t) = \phi(t)\tilde{z}_0$.

Thus, operator L verifies the following proposition:

Proposition 4.1 *The operator L verifies*

i L is a linear operator, continuous and bounded in $\tilde{z}$.

ii $dim\ KerL = codim\ Im\ L = 1$.

iii L is a Fredholm ([6, 10]) operator of index 0 such that :

$$\forall \tilde{z} \in X,\ \exists\ \alpha \in \mathbb{R}, \exists v \in KerL^{\perp},\ \tilde{z} = \alpha \tilde{Z}_0 + v$$

and

$$\forall h \in Y,\ \exists\ h_1 \in ImL,\ \exists\ h_2 \in KerL^{\perp},\ h = h_1 + h_2$$

Remark: By definition, index $L = dim\{KerL\} - codim\{ImL\}$.

Proof :

i. results from the definition of L.

ii. $Ker\ L$ is spanned by $\tilde{Z}_0(t)$, thus $dimKer\ L = 1$, and from the Fredholm Alternative, problem $L\tilde{z} = h$ admits a solution if and only if:

$$\int_0^T < h(t), \tilde{Z}_0(t) > dt = 0, \tag{7}$$

where $< .,. >$ represents the inner product in $\mathbb{R}^{n-1}$. So $codim\ Im\ L = 1$ and $Indix\ L = 0$.

iii. As $Ker\ L$ and $Im\ L$ have, respectively, finite dimension and finite codimension, knowing that they are closed, we can define continuous projections on each one, thus considering:

$$P_0 \quad : X \to Ker\ L$$
$$\tilde{z} \to (P_0\tilde{z})(t) = c(\int_0^T < \tilde{z}(s), \tilde{Z}_0(s) > ds)\tilde{Z}_0(t)$$

where $c = \left(\int_0^T ||\tilde{Z}_0(s)||_0^2 ds\right)^{-2}$, is chosen in order to normalize $P_0\tilde{z}$.

P_0 is a projection on $Ker\ L$, and therefore:

$$\forall \tilde{z} \in X \quad \tilde{z} = \tilde{z}_1 + \tilde{z}_2$$

where $\tilde{z}_1 \in Ker\ L$ and $\tilde{z}_2 \in Ker\ L^{\perp}$ and $\tilde{z}_2$ is such that

$$\int_0^T < \tilde{z}_2(s), \tilde{Z}_0(s) > ds = 0$$

$\triangle$.

Now, we define the projection:

$$\begin{array}{rccl} P_1: & Y & \to & Im\ L \\ & y & \to & (P_1 y)(t) \\ & & & = y(t) - (\int_0^T < y(s), \tilde{Z}_0(s) > ds)\tilde{Z}_0(t) \end{array} \tag{8}$$

and $K : P_1 Y = Im\ L \to (I - P_0)X$ is continuous

We obtain the following lemmas:

[4] L^{-1} is equal to the identity function in this case

Lemma 4.1 *Problem (2) is equivalent to:*

$$L(\alpha\tilde{Z}_0 + v, u_1) = N(\alpha\tilde{Z}_0 + v, u_1) \quad (9)$$

in the sense that each solution of (2) is a solution of (9) and inversely.

Remark : The proof follows immediately from (5) and the previously defined projection.

Lemma 4.2 *$\tilde{z}$ is a solution of Problem (2) if and only if $\tilde{z} = \alpha\tilde{Z}_0 + v$, where $(\alpha, v) \in \mathbb{R} \times (I - P_0)X$ is a solution of :*

$$\begin{cases} v &= KP_1N(\alpha\tilde{Z}_0 + v, u_1) \\ 0 &= (I - P_1)N(\alpha\tilde{Z}_0 + v, u_1) \end{cases} \quad (10)$$

Proof : From the Fredholm alternative, a solution to problem (9) exists if and only if

$$(I - P_0)N = 0$$

So, there exists $\tilde{z} = \tilde{z}(N)$ which is a solution of (9) such that $P_0\tilde{z}(N) = 0$, and the proof follows. $\triangle$

Remark : The first equation of (10) is called the auxiliary equation and the second is the well known bifurcation equation. In fact, this formulation of our problem permits us to transform the resolution of the problem (2) in an infinite dimension to the resolution of two equations : the first one the auxiliary equation are in infinite dimension but with an unique solution $v*$ and the second one are resolved on $\Re$ thus in finite dimension.

4.2.1 Study of the auxiliary equation :

In order to discuss solution of the first part of (10) we introduce a new operator. Let :

$$H : (I - P_0)X \times \mathbb{R} \times U \rightarrow (I - P_0)X$$

be defined by

$$H(v, \alpha, u_1) = v - KP_1N(\alpha\tilde{Z}_0 + v, u_1) = 0 \quad (11)$$

Moreover, H verifies the following assumption :
i) $H(0, 0, 0) = 0$.

ii)H is C^2 with respect to v

iii)$D_vH(., 0, 0)|_{v=0} = Id|_{(I-P_0)X}$ where D_v is the Frechet partial derivative of H relative to v.

In this case the implicit function theorem ensures that equation (11) admits a unique solution $v^*(\alpha, u_1)$ on a neighborhood V_v in $(I - P_0)X$, where $\alpha \in V_\alpha$, with V_α is a neighborhood of α close to 0 in $\mathbb{R}$. Moreover, $u_1 \in V_{u_1}$ where V_{u_1} is a neighborhood of u_1 close to 0 in U. We also have $v^*(0, 0) = 0$ and v^* is continuous and relative to α and u_1 and v^* is defined from $V_\alpha \times V_{u_1} \longrightarrow V_v$. Moreover v^* belongs to class C^2 with respect to α. Thus, we can write v^* as :

$$v^*(\alpha, u_1) = a(u_1) + b(u_1)\alpha + \frac{1}{2}c(u_1)\alpha^2 + d(\alpha, u_1)$$

where $a(u_1)$, $b(u_1)$ and $c(u_1)$ are functions defined from V_{u_1} to V_v and $d(\alpha, u_1)$ is on $O(|\alpha|^2)$.

Remark : As v^* is uniquely determined in V_v, the number of solutions to problem (2) is exactly determined by the number of α solutions in the bifurcation equation, because each solution will be written as :

$$\tilde{z} = \alpha\tilde{Z}_0 + v^*(\alpha, u_1)$$

4.2.2 Bifurcation equation analysis

Now the second equation of (10) the so called Bifurcation equation is equivalent to :

$$\int_0^T < N(\alpha\tilde{Z}_0 + v^*(\alpha, u_1), u_1), \tilde{Z}_0 > dt = 0 \quad (12)$$

Rewriting (12) , we have:

$$\begin{aligned} \bar{I} &:= V_\alpha \times V_{u_1} \rightarrow \Re \\ \bar{I}(\alpha, u_1) &= \int_0^T < \Big(\int_0^t \phi(t)\phi(s)^{-1}\gamma(\alpha\tilde{Z}_0(s) \\ &+ v^*(\alpha, u_1(s)), u_1(s), s)ds \Big), \tilde{Z}_0(t) > dt \\ &= 0 \end{aligned}$$

As $\bar{I}$ is a C^2-function in α, we can write :

$$\bar{I}(\alpha, u_1) = \hat{a}(u_1) + \hat{b}(u_1)\alpha + \frac{1}{2}\hat{c}(u_1)\alpha^2 + \tilde{d}(\alpha, u_1) \quad (13)$$

where $\hat{a}$, $\hat{b}$ and $\hat{c}$ are functions defined from V_{u_1} to $\mathbb{R}$ and $\tilde{d}(\alpha, u_1) = O(|\alpha|^2)$ when $\alpha \rightarrow 0$.

First consideration

Suppose that : $\exists u_1^* \in V_{u_1}$ such that $\frac{\partial \bar{I}}{\partial \alpha}(\alpha, u_1^*) \neq 0$ for $\alpha \in \hat{V}_\alpha$ where $\hat{V}_\alpha \subset V_\alpha$. In this case, the Implicit function theorem ensures that there exists a unique solution $\alpha^* \triangleq \alpha^*(u_1)$ for each $u_1 \in \hat{V}_{u_1}$ where $\hat{V}_{u_1}$ is a neighborhood of u_1^* such that $u_1^* \in \hat{V}_{u_1} \subset V_{u_1}$, and equation $\bar{I}(\alpha^*, u_1^*) = 0$. Thus, we also find that $\alpha^*(0) = 0$ and α^* is continuous on V_{u_1}. So we have no bifurcation here and the nonlinear problem (2) has **a unique** solution $\tilde{z}^* = \alpha^*\tilde{Z}_0 + v^*(\alpha^*, u_1)$ where $\alpha^* \in \hat{V}_\alpha$, $u_1^* \in \hat{V}_{u_1}$ and $v^* \in V_v$ in $(I - P_0)X$. This solution is the continuous extension of the $\tilde{Z_0}(t)$, which is the solution of the associated linear problem.

Second consideration

$$\frac{\partial \bar{I}}{\partial \alpha}(\alpha, u_1) = 0,\ \alpha \in V_\alpha, \qquad u_1 \in V_{u_1}$$

knowing that:

$$\frac{\partial^2 \bar{I}}{\partial \alpha^2}(0, u_1) \neq 0, \quad \forall u_1 \in V_{u_1} \qquad (\gamma \text{ being quadratic in } \tilde{z}).$$

Thus, the Implicit Functions theorem is applied for $\frac{\partial \bar{I}}{\partial \alpha}(\alpha, u_1) = 0$ and therefore there exists a unique solution $\alpha^{**} = \alpha^{**}(u_1)$ such that $\frac{\partial \bar{I}}{\partial \alpha}(\alpha^{**}, u_1) = 0$ with $\alpha^{**}(0) = 0$ and α^{**} is continuous and relative to u_1, u_1 being defined on a neighborhood $\tilde{V}_{u_1} \subset V_{u_1}$ and α^{**} defined from $\tilde{V}_{u_1}$ to $\hat{V}_\alpha \subset V_\alpha$, where $\tilde{V}_\alpha$ is a neighborhood such that $\tilde{V}_\alpha \subset V_\alpha$.

Therefore, we can write :

$$\begin{aligned} \bar{I}(\alpha, u_1) &= \bar{I}(\alpha^{**}, u_1) + \frac{1}{2}\frac{\partial^2 \bar{I}}{\partial \alpha^2}(\alpha^{**}, u_1)(\alpha - \alpha^{**})^2 \\ &+ O(|\alpha - \alpha^{**}|^2) \qquad (14) \\ &= 0 \end{aligned}$$

when $\alpha \longrightarrow \alpha^{**}$.

$\frac{\partial^2 \bar{I}}{\partial \alpha^2}(\alpha^{**}, u_1)$ has the same sign as $\frac{\partial^2 \bar{I}}{\partial \alpha^2}(0, u_1)$ for $\alpha^{**} \in \tilde{V}_\alpha$ and $u_1 \in \tilde{V}_{u_1}$ being sufficiently small.

We know that $\frac{\partial^2 \bar{I}}{\partial \alpha^2}(0, u_1) \neq 0$, so without loss of generality, we can suppose that $\frac{\partial^2 \bar{I}}{\partial \alpha^2}(0, u_1) > 0$. In this case α^{**} is a minimum for $\bar{I}(., u_1)$, and inversely when $\frac{\partial^2 \bar{I}}{\partial \alpha^2}(0, u_1) < 0$, in that case, α^{**} is a maximum for $\bar{I}(., u_1)$.

Remark As $\frac{\partial^2 \bar{I}}{\partial \alpha^2}(0, u_1) \neq 0$, this implied four possibilities :

i- When $\frac{\partial^2 \bar{I}}{\partial \alpha^2}(0, u_1) > 0$ and $\bar{I}(\alpha^{**}, u_1) < 0$ then equation (14) has two solutions in the neighborhood of α^{**}

ii- When $\frac{\partial^2 \bar{I}}{\partial \alpha^2}(0, u_1) < 0$ and $\bar{I}(\alpha^{**}, u_1) > 0$ then equation (14) has two solutions in the neighborhood of α^{**}

iii- When $\frac{\partial^2 \bar{I}}{\partial \alpha^2}(0, u_1) > 0$ and $\bar{I}(\alpha^{**}, u_1) > 0$ then equation (14) has a unique solutions in the neighborhood of α^{**}

iv- When $\frac{\partial^2 \bar{I}}{\partial \alpha^2}(0, u_1) < 0$ and $\bar{I}(\alpha^{**}, u_1) < 0$ then equation (14) has a unique solutions in the neighborhood of α^{**}.

Thus, if we denote by :

$$C(u_1) = \bar{I}(\alpha^{**}(u_1), u_1).\frac{\partial^2 \bar{I}}{\partial \alpha^2}(0, u_1)$$

where the function C is defined from $\tilde{V}_{u_1} \longrightarrow \mathbb{R}$ and

$$\begin{aligned} W' &= \{u_1 \in \tilde{V}_{u_1} / C(U_1) < 0\} \\ W'' &= \{u_1 \in \tilde{V}_{u_1} / C(U_1) = 0\} \\ W''' &= \{u_1 \in \tilde{V}_{u_1} / C(U_1) > 0\} \end{aligned}$$

We then obtain, with respect to control set W', W", W'" for the second consideration :

Theorem 4.2 *There exist, a neighborhood V_v of $v = 0$ in X, a neighborhood $\tilde{V}_\alpha$ of $\alpha = 0$ in $\Re$, a neighborhood $\tilde{V}_{u_1}$ of $u_1 = 0$ in U and a control $u_1 \in \tilde{V}_{u_1}$ such that if :*

1. *$u_1 \in W'$, then problem (2) admits two distinct solutions which bifurcate from $\tilde{Z}_0$.*

2. *$u_1 \in W''$, then problem (2) admits a unique solution in the neighborhood of $\tilde{Z}_0$.*

3. *$u_1 \in W'''$, then problem (2) has no solution in the neighborhood of $\tilde{Z}_0$.*

Remark : Obviously, if $u_1 \in W$" (case 2 of the previous theorem) the problem 2 verify the Cauchy- Lipschitz Conditions [9] or at least the Carathéodory conditions.

5 Conclusion

Our main result, Theorem 4.2, highlights the fact that with respect to the u_1 choice, the driftless integrability of the nonlinear system (2) has different possibilities : one unique solution, two solutions or no solution in the neighborhood of the linear one. The linear solution is generally the one known and computed by the engineer, but it is "pertinent" solution with respect to quadratic uncertainties only if $u_1 \in W$". In the other case, we show a bifurcation with respect to the existence and uniqueness of the solution in the neighborhood of linear one. This type of bifurcation is, to our knowledge, not currently studied in control theory [3] and in control application [4]. In fact, for our system, there exists no solution (in the neighborhood of the linear one) in the case where u_1 is in W''' and consequently u_1 must be taken outside W'''. We think that, in the case where u_1 is in W' as both solutions stay close to the linear one, this may be enough for some control purposes. Thus in this paper, we want just said for system (2) "linearly" approximated, take care to the u_1 choice and this not only with respect to stability or controllability bifurcation but also for existence and uniqueness of real solution close to the approximated one.

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