Quadratic Stabilization of Nonlinear Discrete Time Control Systems with Uncontrollable Linearization

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Abstract

In this paper we study the problem of stabilization of nonlinear systems with uncontrollable linearization. The use of normal forms, permit to find conditions on controllability and stabilizability.

We remark that conditions on controllability and stabilizability are intimately related to a set of quadratic invariants.

Keywords: Nonlinear Systems, Discrete Time Systems, Normal Forms, Controllability, Stabilizability, Bifurcations.

1 Introduction

In this paper, we analyze nonlinear discrete time control systems with one uncontrollable mode, with a single input. For that we use normal forms which permits to transform the systems into its simplest form, and hence the study becomes easier.

The problem is to study controllability and local stabilizability for systems whose dynamics are described by

$$\xi^+ = f(\xi, v) \tag{1}$$

where $\xi^+ = \xi(k+1)$, $\xi \in \mathbb{R}^n$ is the state, $v \in \mathbb{R}$ is the input and $\mu \in \mathbb{R}$ is a parameter. Let us suppose that the systems is not linearly controllable and that there exists only one uncontrollable mode, i.e.

$$\text{rank}\begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix} = n-1 \tag{2}$$

where $A = \dfrac{\partial f}{\partial \xi}(0,0)$, $B = \dfrac{\partial g}{\partial v}(0,0)$.

Let us expand (1) in Taylor series and transform its linear part to the Brunovskỳ form

$$\begin{aligned} z^+ &= (\varepsilon+\lambda)z + f_1^{[2]}(z,x) + g_1^{[1]}(z,x)u + h_1^{[0]}u^2 + O(x,u,z)^3 \\ x^+ &= (I_d+A_2)x + B_2u + f_2^{[2]}(z,x) + g_2^{[1]}(z,x)u + h_2^{[0]}u^2 + O(x,u,z)^3 \end{aligned} \tag{3}$$

with z is the linearly uncontrollable state, x represents the linearly controllable part. λ being the uncontrollable mode of the linearization at $(z,x) = (0,0)$ and $\varepsilon = \pm 1$ representing the limit of stability of linear discrete time systems, and u is the new input. $f_i^{[j]}$ and $g_i^{[j]}$ are polynomials of degree j.

As shown in [9], the quadratic normal forms of such systems are given by

- If $\lambda \notin \{-\varepsilon, 1-\varepsilon\}$

$$\begin{aligned} &z^+ = (\lambda+\varepsilon)z + \textstyle\sum_{i=1}^{n-1} \gamma_{x_i x_i} x_i^2 + \gamma_{z x_1} z x_1 + h_1^{[0]} u^2 + O(x,z,u)^3 \\ &x^+ = (I_d + A_2)x + B_2 u + \bar{f}^{[2]}(x) + \bar{h}_2^{[0]} u^2 + O(x,z,u)^3 \end{aligned} \tag{4}$$

- If $\lambda = 1-\varepsilon$

$$\begin{aligned} &z^+ = z + \textstyle\sum_{i=1}^{n-1} \gamma_{x_i x_i} x_i^2 + \gamma_{zz} z^2 + \gamma_{z x_1} z x_1 + h_1^{[0]} u^2 + O(x,z,u)^3 \\ &x^+ = (I_d + A_2)x + B_2 u + \bar{f}^{[2]}(x) + \bar{h}_2^{[0]} u^2 + O(x,z,u)^3 \end{aligned} \tag{5}$$

- If $\lambda = -\varepsilon$

$$\begin{aligned} &z^+ = \textstyle\sum_{i=1}^{n-1} \{\gamma_{x_i x_i} x_i^2 + \gamma_{z x_i} z x_i\} + \gamma_{zz} z^2 + \theta z u + h_1^{[0]} u^2 + O(x,z,u)^3 \\ &x^+ = (I_d + A_2)x + B_2 u + \tilde{\theta} z u + \bar{f}^{[2]}(x) + \bar{h}_2^{[0]} u^2 + O(x,z,u)^3 \end{aligned} \tag{6}$$

with:

$$\bar{f}^{[2]}(x) = [\bar{f}_1^{[2]}(x)\ \bar{f}_2^{[2]}(x)\ \ldots\ \bar{f}_{n-1}^{[2]}(x)]^T \tag{7}$$

$$\bar{f}_i^{[2]}(x) = \begin{cases} \displaystyle\sum_{j=i+2}^{n} a_{ij} x_j^2 & i = \overline{1, n-2}, \\ 0 & i = n-1. \end{cases} \tag{8}$$

and

$$\bar{h}_2^{[0]} = [h_{2,1}^{[0]} \cdots h_{2,n-2}^{[0]}\, 0]^T$$

The matrices $A_2 \in \mathbb{R}^{(n-1)\times(n-1)}$ and $B_2 \in \mathbb{R}^{(n-1)\times 1}$ are given by:

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Now, we recall theorems in [9] which gives the parametrization for the equilibrium set of systems satisfying assumption (2). Let us recall that an equilibrium set of (1) is given by

$$E = \{\xi | \exists\, u_0 \text{ such that } \xi - f(\xi, u_0) = 0\}$$

Theorem 1.1 *Consider the system (3). If $\lambda \neq 1-\varepsilon$, then, there exists an open U in the neighborhood of $(z,x) = (0,0)$, such that the points in $\bar{E} \cap U$ satisfy:*

$$\begin{aligned} x_1 &= \nu \\ z &= O(\nu)^2 \\ x_i &= O(\nu)^2, \qquad i = \overline{2, n-1}. \end{aligned} \tag{9}$$

$\diamond$

Remark: This Theorem shows that, in a neighborhood of the origin, the equilibrium set is reduced to the origin for a given value of x_1, and so there is no bifurcation.

The topology of the equilibrium set for systems with $\lambda = 1-\varepsilon$, depends on the quadratic part of its normal form. This part will be associated to the matrix:

$$Q = \begin{bmatrix} \gamma_{zz} & \frac{1}{2}\gamma_{zx_1} \\ \frac{1}{2}\gamma_{zx_1} & \gamma_{x_1x_1} \end{bmatrix}$$

The term of intercorrelation between x_1 and z, can be canceled by rotation of the quadratic surface which approximates the equilibrium set, i.e., by diagonalization of the matrix Q. Let T be the diagonalization matrix, suppose that it is orthogonal, i.e. $Q = T^T D T$, D being the diagonal form.

Theorem 1.2 *Given a system (3), with $\lambda = 1-\varepsilon$.*

i. If

$$det(Q) > 0, \tag{10}$$

then, there is no equilibrium point other than

(z,x)=(0,0) near the origin.

ii. If

$$det(Q) < 0, \tag{11}$$

then, the equilibrium set has the following parametrization[1]*:*

$$\begin{aligned} x_i &= O(\nu)^2, \text{ for } i = \overline{2, n-1}. \\ \begin{bmatrix} z \\ x_1 \end{bmatrix} &= T \begin{bmatrix} 1 \\ \pm\sqrt{-\frac{\lambda_1}{\lambda_2}} \end{bmatrix} \nu + O(\nu)^2 \end{aligned} \tag{12}$$

$\diamond$

2 Main Results

2.1 The Controllability

In this section we study the controllability of the system at the equilibrium points in E. Let us note that the controllability we are speaking about is the one associated to the linearization.

Definition 2.1 *Let $(\tilde{x}_0, \nu_0)$ be in E. The system*

$$\tilde{x}^+ = f(\tilde{x}, \nu, u)$$

is linearly controllable at $(\tilde{x}_0, \nu_0)$ if its linearization $(A_{\tilde{x}_0\nu_0}, B_{\tilde{x}_0\nu_0})$ is controllable.

The study of controllability of the points in the neighborhood of the origin depends on the quadratic part. Let us begin with the case where $\lambda \notin \{-\varepsilon, 1-\varepsilon\}$, for that we have the following theorem:

Theorem 2.1 *Consider the system (3), with $\lambda \notin \{-\varepsilon, 1-\varepsilon\}$. If $\gamma_{x_1x_1} \neq 0$, then there exists a neighborhood U of $(z,x) = (0,0)$ such that the system is controllable for all equilibrium points in U except at the origin.* $\diamond$

Proof of Theorem 2.1. Since (3) with $\lambda \notin \{-\varepsilon, 1-\varepsilon\}$ can be transformed into its normal form (4), and since a change of coordinates and a feedback does not affect the controllability of the linearization (since the linear part is invariant), then it is sufficient to study the controllability of the normal form. Let us note the linearization of the normal form (4) around an equilibrium point (9), by:

$$(A_\nu, B_\nu)$$

[1]where $\overline{2, n-1}$ denotes that i is an integer which varies from 2 to $n-1$.

using (4) and (9), we find:

$$
\begin{aligned}
A_\nu &= \begin{bmatrix} \lambda+\varepsilon & 0 \\ 0 & I_d + A_2 \end{bmatrix} \\
&+ \begin{bmatrix} \gamma_{zx_1}\nu & 2\gamma_{x_1x_1}\nu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + O(\nu)^2
\end{aligned}
$$

and

$$B_\nu = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} + O(\nu)^2 \tag{13}$$

It can be proved that the $i-th$ line of $A_\nu^k B_\nu$, noted $\{A_\nu^k B_\nu\}_i$, equals to, for $k = 0, \cdots, n-1$

$$\{A_\nu^k B_\nu\}_i = 2^k \delta_{i,n-k} + O(\nu)^2 \tag{14}$$

and

$$A_\nu^{n-1} B_\nu = \begin{bmatrix} 2^n \gamma_{x_1x_1}\nu & 0 & \cdots & 0 \end{bmatrix} + O(\nu)^2 \tag{15}$$

using (14) and (15) the matrix of controllability have full rank for small variations of ν if $\gamma_{x_1x_1} \neq 0$. ∎

Remark: The same theorem is valid for the case where $\lambda = -\varepsilon$. The proof is slightly different, since the normal form changes. For the sake of brevity, We omit it.

For $\lambda = 1 - \varepsilon$, the topology of the equilibrium set change. The following Theorem permits the study of the controllability of such systems.

Theorem 2.2 *Consider the system (3), with $\lambda = 1 - \varepsilon$ and suppose the inequality (11) satisfied. If the coefficients of the resonant terms of the system satisfy:*

$$[\gamma_{zx_1} \; 2\gamma_{x_1x_1}] T \begin{bmatrix} 1 \\ \pm\sqrt{-\frac{\lambda_1}{\lambda_2}} \end{bmatrix} \neq 0 \tag{16}$$

then, the system is linearly controllable for the equilibrium points in the neighborhood of the origin in $E_\pm \backslash \{(0,0)\}$. ◇

Proof of Theorem 2.2. As in the precedent theorems we use (5), its linearization around an equilibrium point $(z,x) \in E$ is:

$$
\begin{aligned}
A_\nu &= \begin{bmatrix} 1 & 0 \\ 0 & I_d + A_2 \end{bmatrix} \qquad (17) \\
&+ \begin{bmatrix} \chi_1 & \chi_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + O(\nu)^2
\end{aligned}
$$

with $\chi_1 = \gamma_{zx_1}x_1 + 2\gamma_{zz}z$ and $\chi_2 = 2\gamma_{x_1x_1}x_1 + \gamma_{zx_1}z$, and B_ν is given by (13).

It can be proved, that the $i-th$ component of $A_\nu^k B_\nu$ is given by (14).

Moreover:

$$A_\nu^{n-1} B_\nu = \begin{bmatrix} 2^{n-1}(2\gamma_{x_1x_1}x_1 + \gamma_{zx_1}z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} + O(\nu)^2 \tag{18}$$

Consequently for an equilibrium point defined by (12) and with (18) we obtain:

$$A_\nu^{n-1} B_\nu = \begin{bmatrix} 2^{n-1}\varpi \\ 0 \\ \vdots \\ 0 \end{bmatrix} + O(\nu)^2 \tag{19}$$

with $\varpi = [\gamma_{zx_1} \; 2\gamma_{x_1x_1}] T \begin{bmatrix} 1 \\ \pm\sqrt{-\frac{\lambda_1}{\lambda_2}} \end{bmatrix}$.

We find from (14) and (19) that the controllability matrix $[B_\nu, A_\nu B_\nu, \cdots, A_\nu^{n-1}B_\nu]$ has a full rank for small values of ν if $\varpi \neq 0$.

Hence:

1. if $[\gamma_{zx_1} \; 2\gamma_{x_1x_1}] T \begin{bmatrix} 1 \\ \sqrt{-\frac{\lambda_1}{\lambda_2}} \end{bmatrix} \neq 0$, then the system is linearly controllable in $E_+ \backslash \{0,0\}$.
2. if $[\gamma_{zx_1} 2\gamma_{x_1x_1}] T \begin{bmatrix} 1 \\ -\sqrt{-\frac{\lambda_1}{\lambda_2}} \end{bmatrix} \neq 0$, then the system is linearly controllable in $E_- \backslash \{0,0\}$. ∎

Remark: if $det(Q) > 0$, the system is not linearly controllable, since the equilibrium set is reduced to $(z,x) = (0,0)$, which is not linearly controllable.

2.2 The Stabilizability

Now, we find sufficient conditions for the stabilizability of controlled systems using the coefficients of resonant terms in the normal forms. Starting from (3), and for $|\lambda + \varepsilon| > 1$, the system is not stabilizable by a class C^1 feedback. On the other hand for $|\lambda + \varepsilon| < 1$, the system is stabilizable.

The case $|\lambda + \varepsilon| = 1$ is critical. This case is treated using Poincaré analysis and the theorem of center manifold.

Theorem 2.3 *Consider the system (3), with $|\lambda+\varepsilon| = 1$. If $\gamma_{zx_1} \neq 0$, then there exists a quadratic controller which asymptotically stabilizes the origin.* ◇

Proof of Theorem 2.3 . At first, we remark that the case $|\lambda+\varepsilon| = 1$ is subdivided into :

1. $\lambda+\varepsilon = -1$, this case corresponds to the normal form (4).
2. $\lambda+\varepsilon = 1$, this case corresponds to the normal form(5).

then, according to (4) and (5) we see that there is only one term which distinguishes the two cases, it is $\gamma_{zz} z^2$. By combining the two normal forms, we see that z^+ in (4) and (5) can be written as

$$\begin{aligned} z^+ &= (\lambda+\varepsilon)\, z + \sum_{i=1}^{n-1} \gamma_{x_i x_i}\, x_i^2 + \gamma_{z x_1} z\, x_1 \\ &+ \frac{\lambda+\varepsilon+1}{2}\, \gamma_{zz}\, z^2 + h_1^{[0]}\, u^2 + O(z,x,u)^3 \end{aligned} \quad (20)$$

The $x-$part is common and equals to

$$x^+ = (I_d + A_2)x + B_2 u + \tilde{f}^{[2]}(x) + \bar{h}_2^{[0]} u^2 + O(x,z,u)^3 \quad (21)$$

Since normalization does not affect linear controllability, then we will prove the theorem for the normal form Consider the feedback law:

$$u(z,x_1) = F_1 x_1 + F_2 x_2 + \cdots + F_{n-1}x_{n-1} + \sigma z + \mu z^2 \quad (22)$$

with $F = [F_1, F_2, \cdots, F_{n-1}]^T$ stabilizing the linearly controllable part, i.e. $A_2 + B_2 F$ is Hurwitz, and $F_1 \neq 0$.

From the previous argument, and since the z-part have a critical eigenvalue. Hence, to analyze this case, it is sufficient to use the center manifold theorem [7].

The center manifold of the closed loop system is given by:

$$x = \pi(z) = \begin{bmatrix} \pi_1(z) \\ \vdots \\ \pi_{n-1}(z) \end{bmatrix} = \alpha z + \beta z^2 + O(z^3) \quad (23)$$

with $\alpha = [\alpha_1, \ldots, \alpha_{n-1}]^T$ and $\beta = [\beta_1, \ldots, \beta_{n-1}]^T$. The coefficients being unknown, we should find equations that they satisfy. Injecting the expression (23) in (22), we obtain

$$u = (F\alpha + \sigma)z + (F\beta + \mu)z^2 + O(z^3) \quad (24)$$

Injecting (24) in (20) , we obtain

$$\begin{aligned} z^+ &= (\lambda+\varepsilon)z + \Big[\sum_{i=1}^{n-1} \gamma_{x_i x_i}\alpha_i^2 + \gamma_{z x_1}\alpha_1 \quad (25) \\ &+ h_1^{[0]}(\sigma + F\alpha)^2 + \frac{\lambda+\varepsilon+1}{2}\gamma_{zz}\Big] z^2 + O(z^3) \end{aligned}$$

Moreover, since we are on the center manifold $x = \pi(z)$, then using (23) and (25)

$$\begin{aligned} \pi(z^+) &= (\lambda+\varepsilon)\alpha z + \Big[\big(\sum_{i=1}^{n-1} \gamma_{x_i x_i}\alpha_i^2 + \gamma_{z x_1}\alpha_1 + \beta \\ &+ h_1^{[0]}(\sigma + F\alpha)^2 + \frac{\lambda+\varepsilon+1}{2}\gamma_{zz}\big)\alpha\Big] z^2 + O(z^3) \end{aligned} \quad (26)$$

Replacing (24) in (21), we obtain:

$$\begin{aligned} x^+ &= \Big\{(I_d + A_2)\alpha + B_2(F\alpha + \sigma)\Big\} z + \Big\{(I_d + A_2)\beta \\ &+ (F\beta + \mu)B_2 + h_2^{[0]}(F\alpha + \sigma)^2\Big\} z^2 + O(z^3) \end{aligned} \quad (27)$$

Since $x^+ = \pi(z^+)$, and using (26), (27), we obtain the following system of equations:

$$[((1 - \lambda - \varepsilon)\, I_d + A_2) + B_2 F]\alpha + B_2\sigma = 0$$

$$\begin{aligned} &(A_2 + B_2 F)\beta + B_2\mu + h_2^{[0]}(F\alpha + \sigma)^2 = \\ &\Big[\sum_{i=1}^{n=1} \gamma_{x_i x_i}\alpha_i^2 + \gamma_{z x_1}\alpha_1 + h_1^{[0]}(\sigma + F\alpha)^2 + \frac{\lambda+\varepsilon}{2}\,\gamma_{zz}\Big]\alpha \end{aligned}$$

Hence, to find α_i and β_i for $i = \overline{1, n-1}$, it is sufficient to resolve the precedent system of equations.

1. Determination of the α_i, $i = \overline{1, n-1}$:

   Using (2.2), and the expressions of A_2 and B_2, we obtain the following system of equations:

$$\begin{aligned} (1 - \lambda - \varepsilon)\alpha_1 + \alpha_2 &= 0 \\ &\vdots \\ (1 - \lambda - \varepsilon)\alpha_{n-2} + \alpha_{n-1} &= 0 \\ (F_{n-1} + (1 - \lambda - \varepsilon))\alpha_{n-1} + \sum_{i=1}^{n-2} F_i \alpha_i &= -\sigma \end{aligned} \quad (28)$$

   Using the first $n-2$ equations of (28), we obtain:

$$\alpha_i = (\lambda + \varepsilon - 1)^{i-1}\alpha_1 \qquad i = \overline{2, n-1}$$

   α_1 is obtained by replacing this expression in (28):

$$\alpha_1 = -\frac{\sigma}{(\lambda+\varepsilon-1)^{n-1} + \sum_{i=1}^{n-1}(\lambda+\varepsilon-1)^{i-1}F_i} \quad (29)$$

   if $(\lambda+\varepsilon-1)^{n-1} + \sum_{i=1}^{n-1}(\lambda+\varepsilon-1)^{i-1}F_i = 0$, then $\sigma = 0$ and α_1 can take any real value.

2. Determination of the β_i, $i=\overline{1,n-1}$:

Using (28), the expressions of A_2 and B_2, we obtain:

$$\begin{aligned}\beta_1 &= \frac{1}{F_1}\Big\{-\mu-\sum_{i=2}^{n-1}F_i\beta_i-h^{[0]}_{2,n-1}(F\alpha+\sigma)^2\\ &+ [\sum_{i=1}^{n-1}\gamma_{x_ix_i}\alpha_i^2+\gamma_{zx_1}\alpha_1+h_1^{[0]}(\sigma+F\alpha)^2\\ &+ \frac{\lambda+\varepsilon+1}{2}\gamma_{zz}]\alpha_{n-1}\Big\}\end{aligned}$$

$$\begin{aligned}\beta_i &= -h^{[0]}_{2,i-1}(F\alpha+\sigma)^2+\Big\{\sum_{i=1}^{n-1}\gamma_{x_ix_i}\alpha_i^2+\gamma_{zx_1}\alpha_1\\ &+ h_1^{[0]}(\sigma+F\alpha)^2+\frac{\lambda+\varepsilon+1}{2}\gamma_{zz}\Big\}\alpha_{i-1}\end{aligned}$$

for $i=2,\cdots,n-1$.

Using the expressions of α_i and β_i for $i=\overline{1,n-1}$, we can write the projection of the closed loop system on the center manifold:

$$\begin{aligned}z^+ &= (\lambda+\varepsilon)z+\textstyle\sum_{i=1}^{n-1}\gamma_{x_ix_i}x_i^2+\gamma_{zx_1}zx_1+h_1^{[0]}u^2+O(x,z,u)^3\\ &= (\lambda+\varepsilon)z+\Big[\tfrac{\lambda+\varepsilon+1}{2}\gamma_{zz}+\textstyle\sum_{i=1}^{n-1}\gamma_{x_ix_i}\alpha_i^2+\gamma_{zx_1}\alpha_1\\ &+ h_1^{[0]}(\sigma+F\alpha)^2\Big]z^2+\Big[\textstyle\sum_{i=1}^{n-1}\gamma_{x_ix_i}\alpha_i\beta_i+\tfrac{1}{2}\beta_1\gamma_{zx_1}\\ &+ h_1^{[0]}(\sigma+F\alpha)(F\beta+\mu)\Big]z^3+O(z^4)\end{aligned}\tag{30}$$

To have asymptotic stability of (30), we choose σ and μ such that

$$\frac{\lambda+\varepsilon+1}{2}\gamma_{zz}+\sum_{i=1}^{n-1}\gamma_{x_ix_i}\alpha_i^2+\gamma_{zx_1}\alpha_1+h_1^{[0]}(\sigma+F\alpha)^2=0\tag{31}$$

and:

$$(\lambda+\varepsilon)\Big\{\tfrac{1}{2}\beta_1\gamma_{zx_1}+\textstyle\sum_{i=1}^{n-1}\gamma_{x_ix_i}\alpha_i\beta_i+h_1^{[0]}(F\alpha+\sigma)(F\beta+\mu)\Big\}<0\tag{32}$$

Let us note the left hand side of (31) by $\Upsilon(\sigma)$. Then,

$$\begin{aligned}\Upsilon(\sigma) &= \gamma_{zx_1}\alpha_1+\Big\{\sum_{i=1}^{n-1}\gamma_{x_ix_i}(\lambda+\varepsilon-1)^{2i-2}\\ &+ h_1^0(\lambda+\varepsilon-1)^{2n}\Big\}\alpha_1^2+\frac{\lambda+\varepsilon+1}{2}\gamma_{zz}\end{aligned}\tag{33}$$

Using (33), the set of solutions of $\Upsilon(\sigma)=0$, noted as Σ^*, will be given by:

- If $\lambda+\varepsilon=-1$,

$$\sigma\in\left\{0,\ \frac{4\,\gamma_{zx_1}}{\sum_{i=1}^{n-1}\gamma_{x_ix_i}4^i+\sum_{j,k=1}^{n-1}F_jF_k(-2)^{j+k}}\right\}\tag{34}$$

- If $\lambda+\varepsilon=1$, then

$$\begin{aligned}\sigma &= \frac{\gamma_{zz}}{\gamma_{zx_1}}F_1 && \text{if } \gamma_{x_1x_1}=0\\ \sigma &= \frac{\gamma_{zx_1}\pm\sqrt{\gamma_{zx_1}^2-4\gamma_{x_1x_1}\gamma_{zz}}}{2\gamma_{x_1x_1}}F_1 && \text{if } \gamma_{x_1x_1}\neq 0\end{aligned}\tag{35}$$

For μ we use (32),(31),(30) and (30), to obtain :

$$\begin{aligned}\frac{1}{F_1}(\frac{1}{2}\gamma_{zx_1}+\gamma_{x_1x_1}\alpha_1)\mu &> \Big\{\frac{1}{F_1}(\frac{1}{2}\gamma_{zx_1}+\gamma_{x_1x_1}\alpha_1)\,\zeta_1\\ &+ \sum_{i=2}^{n-1}\gamma_{x_ix_i}\alpha_i\beta_i-h_1^{[0]}h^{[0]}_{2,n-1}(F\alpha+\sigma)^3\Big\}\end{aligned}\tag{36}$$

with $\zeta_1=\sum_{i=2}^{n-1}F_i\beta_i+h^{[0]}_{2,n-1}(F\alpha+\sigma)^2$ We conclude that choosing σ as in(34) or (35) and μ given by (36), guarantees the existence of controllers such that projection of the closed loop dynamics on the center manifold is asymptotically stable. Hence, using the center manifold theorem the global closed loop system is asymptotically stable. ∎

3 Conclusion

In this paper, the analysis and the control of nonlinear discrete time systems with one dimensional uncontrollable mode, was done. We used normal forms to study the controllability and the stabilizability in the neighborhood of equilibrium sets. The study of the stabilizability of the system, permitted to propose a quadratic stabilizing controller.

Another relevant topic concerns the analysis and the control of the same class of systems when it is parametrized. In this case we will have two sources of bifurcations. The first is due to the fact that the system is not linearly controllable, as shown in [9] and in this paper. The second source is due to parameters. This work will be presented in another paper.

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