The Dynamic Controller Design Via Quickest Descent Control Method for Nonlinear Systems

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Abstract: We propose a kind of dynamic controller called the quickest descent controller that can be used for general nonlinear systems. The structure of the controller is simple. Both the stability and the performance behavior are considered in the controller design. The theory of feedback passivity for nonlinear systems is used to analyze the stability of the proposed dynamic controller.

Keywords: nonlinear control, dynamic controller, feedback passivity, asymptotical stability.

1 Introduction

Many papers have been published concerning on stabilizing static feedback control laws for nonlinear systems [1],[7],[2], but most of them can only be used effectively for affine nonlinear system. Although some studies were extended to general nonlinear systems, iterative calculations are usually needed to get the control input [2],[7]. More restrictively, prior knowledge of Lyapunov function or control Lyapunov function is necessary. Further more, most of these papers pay little attention to considerations of transient performance of the systems. Although a systematic method is given in [2] to design a stabilizing inverse optimal control law based on prior knowledge of control Lyapunov function, such controller can not be implemented on-line for general nonlinear systems.

It is well known that the model predictive control method can be used to calculate control input on-line, and its objective is to get a better transient performance. But unfortunately, it is very difficult to guarantee the stability of this method, and also, on-line calculation is still a problem for general nonlinear systems.

In this paper, we propose a kind of dynamic controller called the *quickest descent controller* that can be used for general nonlinear systems. It can be constructed even if we do not have any prior knowledge of Lyapunov function. This dynamic controller is developed by using a simple method with clear physical meaning, and its structure is simple enough to be implemented on-line. The main idea of our method is to modify the control input directly at each moment, so that a performance index is decreased by the quickest descent method. A similar form of dynamic controller is developed by using a method of functional analysis in [4].

In order to analyze stability of the systems, we apply the theory of feedback passivity and cascade based design for nonlinear systems. We show that for some special nonlinear systems such as affine nonlinear system, the proposed dynamic controller can guarantee asymptotical stability, while for general nonlinear systems, some modifications to the dynamic controller must be made so that asymptotical stability can be obtained.

2 The Quickest Descent Controller

Our dynamic controller has a close relation with the one proposed in [4]. In order to explain our idea, let us restate briefly the main points in [4]. Consider a nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{2.1}$$

where $\boldsymbol{x} \in R^n$ is the state vector and $\boldsymbol{u} \in R^r$ is the control vector. The aim of control is to decrease a performance index $F(\boldsymbol{x}, \boldsymbol{u})$ at each moment, and then the problem can be written as

$$\begin{aligned} &\underset{\boldsymbol{u}}{\text{decrease}}\ F(\boldsymbol{x}, \boldsymbol{u}) \\ &\text{subj. to } \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{aligned} \tag{2.2}$$

In order to decrease $F(\boldsymbol{x}, \boldsymbol{u})$ at every moment with control $\boldsymbol{u}$, the steepest descent method is used. To do so, we have to calculate a gradient of $F(\boldsymbol{x}, \boldsymbol{u})$ with respect to $\boldsymbol{u}$. Define $\phi^t[\boldsymbol{u}] \triangleq F(\boldsymbol{x}, \boldsymbol{u})$ for the fixed t, then the gradient can be given

$$\nabla \phi^t[\boldsymbol{u}] = \boldsymbol{f}_{\boldsymbol{u}}^T(\boldsymbol{x}, \boldsymbol{u}) F_{\boldsymbol{x}}^T(\boldsymbol{x}, \boldsymbol{u}) + F_{\boldsymbol{u}}^T(\boldsymbol{x}, \boldsymbol{u}) \tag{2.3}$$

at the final time t. Finally the dynamic controller can be obtained as

$$\dot{\boldsymbol{u}} = -\mathcal{L} \nabla \phi^t[\boldsymbol{u}] \tag{2.4}$$

where $\mathcal{L} \triangleq \text{diag}\{\alpha_1, \cdots, \alpha_r\}$, and $\alpha_i > 0$ are positive ccoefficients. It should be noted that in [4] the performance index $F(\boldsymbol{x}, \boldsymbol{u})$ includes explicitly both $\boldsymbol{x}$ and $\boldsymbol{u}$ and the authors regarded $\boldsymbol{x}$ as a function of $\boldsymbol{u}$. The mathematical tool used in [4] is the sensitive analysis for differential equation in the functional space.

The main idea of constructing the dynamic controller (2.4) is to decrease the performance index $F(\boldsymbol{x}, \boldsymbol{u})$ at each moment in a steepest descent direction locally. In this paper, we derive our dynamic controller similar to (2.4) in a different way.

$$\boldsymbol{x}(k+1) = \boldsymbol{f}(\boldsymbol{x}(k), \boldsymbol{u}(k)) \tag{2.5}$$

where $\boldsymbol{x} \in R^n, \boldsymbol{u} \in R^r$ and $\boldsymbol{f}$ is smooth mapping with $\boldsymbol{f}(0,0) = 0$. At any time step k, for a given $\boldsymbol{x}(k)$ and $\boldsymbol{u}(k)$, we can calculate $\boldsymbol{x}(k+1)$. From the consideration of stability, it is naturally trying to find such $\boldsymbol{u}(k)$ that $\boldsymbol{x}(k+1)$ is forced to approach equilibrium as much as possible. This can be done by minimizing a proper performance index as $F(\boldsymbol{x}(k))$. It should be noted that at every time step k, the control action $\boldsymbol{u}(k)$ do not have any effect on $\boldsymbol{x}(k)$, what it effects is the value at next time step $\boldsymbol{x}(k+1)$. That is to say, at any time step k, $\boldsymbol{x}(k)$ is given. What we are trying to do is to find such $\boldsymbol{u}(k)$ so that $F(\boldsymbol{x}(k+1))$ is minimized, or equivalently, to minimize $\frac{F(\boldsymbol{x}(k+1)) - F(\boldsymbol{x}(k))}{\Delta t}$, where Δt is time interval. Such idea can be naturally used to the continuous nonlinear systems.

For continuous nonlinear system (2.1), similar to the performance index F for discrete time system, we define a descent function $W(\boldsymbol{x})$ being a *distance* from a point $\boldsymbol{x}$ to the equilibrium 0. Here, without loss of generality, we suppose that the equilibrium of the system is 0. With same idea for discrete time case (let $\Delta \to 0$), it is clear that we should try to find such control $\boldsymbol{u}$ that decreases $\dot{W}(\boldsymbol{x})$ as much as possible at every point $\boldsymbol{x}$ along a trajectory of nonlinear system (2.1). Therefore the problem becomes $\min_{\boldsymbol{u}} \dot{W}(\boldsymbol{x})$. This is why we call our dynamic controller a quickest descent controller. Different from the considerations in [4], at every point $\boldsymbol{x}$, $\boldsymbol{u}$ is a decision variable and *independent* on $\boldsymbol{x}$ and the gradient of $\dot{W}(\boldsymbol{x})$ with respect to $\boldsymbol{u}$ can be easily calculated as $\dot{W}_{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{f}_{\boldsymbol{u}}^T(\boldsymbol{x}, \boldsymbol{u}) W_{\boldsymbol{x}}^T(\boldsymbol{x})$.

For function $W(\boldsymbol{x})$, suppose the following properties hold: 1. $W(0) = 0$. 2. $W(\boldsymbol{x}) > 0 \quad \forall \boldsymbol{x} \neq 0$. 3. $\nabla W(\boldsymbol{x}) \neq 0 \quad \forall \boldsymbol{x} \neq 0$. It should be noted that the descent function $W(\boldsymbol{x})$ usually is not a Lyapunov function. In section 3, when we analyze the stability of proposed dynamic controller, $W(\boldsymbol{x})$ is also called the Lyapunov-like function. Now our problem becomes as follows

$$\min_{\boldsymbol{u}} \dot{W}(\boldsymbol{x}) \tag{2.6a}$$

$$\text{subj. to } \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{2.6b}$$

Until now the control $\boldsymbol{u}$ in (2.6) is just trying to make $W(\boldsymbol{x})$ approach zero as much as possible, and therefore forces the point $\boldsymbol{x}$ to the equilibrium without considerations of control effort. The similar formulation in the form of (2.6) can also be found in [6] with different considerations.

Next we consider the following problem

$$\begin{aligned} \min_{\boldsymbol{u}} F(\dot{W}(\boldsymbol{x}), \boldsymbol{u}) &\triangleq \dot{W}(\boldsymbol{x}) + \frac{1}{2}\boldsymbol{u}^T R \boldsymbol{u} \\ \text{subj. to } \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned} \tag{2.7}$$

Here we add the item $\frac{1}{2}\boldsymbol{u}^T R \boldsymbol{u}$ in the performance index to penalize the large control action. Actually any meaningful performance index should include such a penalty to control action. For our problem, if we can guarantee that $\dot{W}(\boldsymbol{x}) < 0$ at everypoint $\boldsymbol{x}$, then among all possible choices of $\boldsymbol{u}$, the one with minimum norm is just *inverse optimal* [2]. As to the exact relation between optimal performance and our dynamic controller, we will continue studies in another paper.

If we consider only one step of optimization at every point $\boldsymbol{x}$ for problem (2.7), we get a dynamic controller as follows

$$\begin{aligned} \dot{\boldsymbol{u}} &= -\mathcal{L}\left[\frac{\partial F(\dot{W}(\boldsymbol{x}), \boldsymbol{u})}{\partial \boldsymbol{u}}\right] \\ &= -\mathcal{L}\left[\boldsymbol{f}_{\boldsymbol{u}}^T(\boldsymbol{x}, \boldsymbol{u}) W_{\boldsymbol{x}}^T(\boldsymbol{x}) + R\boldsymbol{u}\right] \end{aligned} \tag{2.8}$$

where $\mathcal{L}$ is a positive diagonal matrix as defined in (2.4). Note for multiple input system, we can expect a better convergence behavior to consider $\mathcal{L}$ as a constant diagonal positive matrix. While for simplicity, $\mathcal{L}$ can often be considered as a constant positive scalar.

Obviously, the structure of this controller is very simple and we can construct the dynamic controller (2.8) for general nonlinear systems without prior knowledeg of Lyapunov function. We can also see that both the stability and the performance is considered in the controller design.

Combining (2.1) and (2.8), we have

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{2.9a}$$

$$\dot{\boldsymbol{u}} = -\mathcal{L}\left[\boldsymbol{f}_{\boldsymbol{u}}^T(\boldsymbol{x}, \boldsymbol{u}) W_{\boldsymbol{x}}^T(\boldsymbol{x}) + R\boldsymbol{u}\right] \tag{2.9b}$$

3 Stability Analysis

In this section, we analyze the stability of control system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{3.1a}$$

$$\dot{\boldsymbol{u}} = -\mathcal{L}\left[\boldsymbol{f}_{\boldsymbol{u}}^T(\boldsymbol{x}, \boldsymbol{u}) W_{\boldsymbol{x}}^T(\boldsymbol{x}) + R\boldsymbol{u}\right] \tag{3.1b}$$

where $W(\boldsymbol{x})$ is the descent function that is regarded as a Lyapunov-like function here. We have explained that such a controller comes from the considerations of both stability and performance for a general nonlinear system. Although simulations of many kinds of nonlinear systems show that such controller works well, we can not guarantee the asymptotical stability in general. In this section, by using the cascade passivity based design method, we show how to ensure the asymptotical stability of the system (3.1).

First, we review some basic definitions and theorem on dissipativity and passivity. For more detail, please refer to [8], [1], [5] and [3]. We will use them to analyze the stability of control system (3.1).

Consider the state space system

$$\Sigma: \quad \begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \end{cases} \tag{3.2}$$

together with a function $s(\boldsymbol{x}, \boldsymbol{y}) \in R$ called the *supply rate*, where $\boldsymbol{x} \in R^n, \boldsymbol{u} \in R^r$ and $\boldsymbol{y} \in R^p$.

Definition 3.1. System Σ is said to be *passive* if it is dissipative with supply rate $s(\boldsymbol{u}, \boldsymbol{y}) = \boldsymbol{u}^T\boldsymbol{y}$. That is, if there exists a function $S(\boldsymbol{x}) \in R^+$, called the storage function, such that for all $\boldsymbol{x}_0$, all $t_1 \geq t_0$, and all input functions $\boldsymbol{u}$

$$S(\boldsymbol{x}(t_1)) \leq S(\boldsymbol{x}(t_0)) + \int_{t_0}^{t_1} \boldsymbol{u}^T \boldsymbol{y} dt \tag{3.3}$$

where $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, and $\boldsymbol{x}(t_1)$ is the state of Σ at time t_1 resulting from initial condition $\boldsymbol{x}_0$ and input function $\boldsymbol{u}(\cdot)$. Then the system Σ is said to be *passive*. Further more, if $S(\boldsymbol{x})$ is C^1, (3.3) becomes

$$\dot{S}(\boldsymbol{x}) \leq \boldsymbol{u}^T\boldsymbol{y} \tag{3.4}$$

Note that for the passive systems, $\boldsymbol{u}$ and $\boldsymbol{y}$ must have the same dimensions.

Definition 3.2. Σ is zero-state observable if $\boldsymbol{u}(t) = 0, \boldsymbol{y}(t) = 0, \forall t \geq 0$, implies $\boldsymbol{x}(t) = 0, \forall t \geq 0$. Σ is zero-state detectable if $\boldsymbol{u}(t) = 0, \boldsymbol{y}(t) = 0, \forall t \geq 0$, implies $\lim_{t\to\infty} \boldsymbol{x}(t) = 0$.

Theorem 3.1 (Theorem 3.2 in [1]). *Suppose Σ is passive with a C^1 storage function $S(\boldsymbol{x})$ which is positive definite. Suppose Σ is locally zero-state detectable. Let $\boldsymbol{\phi} : Y \to U$ be any smooth function such that $\boldsymbol{\phi}(0) = 0$ and $\boldsymbol{y}^T\boldsymbol{\phi}(\boldsymbol{y}) > 0$ for each nonzero y. The control law*

$$\boldsymbol{u} = -\boldsymbol{\phi}(\boldsymbol{y}) \tag{3.5}$$

asymptotically stabilizes the equilibrium $\boldsymbol{x} = 0$. If Σ is zero state detectable and S is proper, the control law (3.5) *globally asymptotically stabilizes the equilibrium $\boldsymbol{x} = 0$.*

Note that in [1], Σ is only considered as affine nonlinear system. But the proof for this theorem can also be applied for general nonlinear systems. Moreover, we are only interested in a special choice of $\boldsymbol{\phi}$, that is, $\boldsymbol{\phi} = -K\boldsymbol{y}$, where K is a positive definite matrix, Accordingly we have $\boldsymbol{u} = -K\boldsymbol{y}$

We need the following assumptions.

Assumption 3.1. *Consider a nonlinear system*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{3.6}$$

and $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is factorized as

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{x}, 0) + G(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{u} \tag{3.7}$$

where $G(\boldsymbol{x}, \boldsymbol{u})$ is a smooth matrix function with dimension $n \times r$. The equilibrium $\boldsymbol{x} = 0$ of unforced part $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, 0)$ is globally stable and a C^2 positive definite proper function $V(\boldsymbol{x})$ is known such that $V_{\boldsymbol{x}}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}, 0) \leq 0$.

Assumption 3.2. *The following relation holds*

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, 0) \\ G^T(\boldsymbol{x}, \boldsymbol{u})|_{\boldsymbol{u}=0} V_{\boldsymbol{x}}^T(\boldsymbol{x}) = 0 \end{cases} \Rightarrow \lim_{t\to\infty} \boldsymbol{x}(t) = 0 \tag{3.8}$$

Remark 3.1. With assumption 3.1 and 3.2, following we discuss how to gurantee the *asymptotically stability* of the system (3.1). Note that in general, we need that the equilibrium $\boldsymbol{x} = 0$ of the unforced part $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, 0)$ is globally stable, but not globally asymptotically stable. In other words, even if the unforced system is only stable but not asymptotical stable, we still have a chance to check the assumption 3.2 to determine if the cascade system (3.1) is asymptotically stable or not.

Remark 3.2. The assumptions 3.1 and 3.2 play an important role in this paper. It should be noted that under the assumption 3.1, the effort to find a feedback control $\boldsymbol{u}$ that can guarantee the globally asymptotical stability of system (3.6) is still meaningful. Actually for many nonlinear systems, even the unforced part is stable or asymptotically stable, with little external disturbance, the state variables of the system go to infinite within finite time. Therefore to design a stabilizing feedback controller for (3.6) is necessary. And more important, we should find a feedback control law $\boldsymbol{u}(\boldsymbol{x})$ rather than $\boldsymbol{u} = 0$ for getting a better transient performance.

From the following theorem, we see that in order to guarantee the globally asymptotical stability for (3.1), additional artificial control $\boldsymbol{v}$ must be introduced into the dynamic controller.

Theorem 3.2. *Consider the following cascade system*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{3.9a}$$

$$\dot{\boldsymbol{u}} = -\mathcal{L}\left[\boldsymbol{f}_{\boldsymbol{u}}^T(\boldsymbol{x}, \boldsymbol{u}) W_{\boldsymbol{x}}^T(\boldsymbol{x}) + R\boldsymbol{u}\right] + \boldsymbol{v} \tag{3.9b}$$

where $\mathcal{L}$ is a positive diagonal constant matrix and $\boldsymbol{v}$ is the artificial control vector introduced to guarantee stability. Suppose that the Assumption 3.1 and 3.2 hold. Then (3.9) *can be made globally asymptotically stable of the equilibrium* $(\boldsymbol{x},\boldsymbol{u})=(\boldsymbol{x}_s,\boldsymbol{u}_s)$ *by using*

$$\begin{aligned}\boldsymbol{v}=&-\boldsymbol{K}\boldsymbol{u}+\mathcal{L}\left[\boldsymbol{f}_{\boldsymbol{u}}^T(\boldsymbol{x},\boldsymbol{u})W_{\boldsymbol{x}}^T(\boldsymbol{x})+R\boldsymbol{u}\right]\\&-G^T(\boldsymbol{x},\boldsymbol{u})V_{\boldsymbol{x}}^T(\boldsymbol{x})\end{aligned}\tag{3.10}$$

Proof. First let

$$\boldsymbol{a}(\boldsymbol{x},\boldsymbol{u})\triangleq-\mathcal{L}\left[\boldsymbol{f}_{\boldsymbol{u}}^T(\boldsymbol{x},\boldsymbol{u})W_{\boldsymbol{x}}^T(\boldsymbol{x})+R\boldsymbol{u}\right]$$

Then (3.9b) becomes $\dot{\boldsymbol{u}}=\boldsymbol{a}(\boldsymbol{x},\boldsymbol{u})+\boldsymbol{v}$

With $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})=\boldsymbol{f}(\boldsymbol{x},0)+G(\boldsymbol{x},\boldsymbol{u})\boldsymbol{u}$ and take $\boldsymbol{y}=\boldsymbol{u}$, $S(\boldsymbol{x},\boldsymbol{u})=V(\boldsymbol{x})+\frac{1}{2}\boldsymbol{u}^T\boldsymbol{u}$, we can verify that

$$\begin{aligned}&\dot{S}(\boldsymbol{x},\boldsymbol{u})=\dot{V}(\boldsymbol{x})+\boldsymbol{u}^T\dot{\boldsymbol{u}}\\&=V_{\boldsymbol{x}}(\boldsymbol{x})[\boldsymbol{f}(\boldsymbol{x},0)+G(\boldsymbol{x},\boldsymbol{u})\boldsymbol{u}]+\boldsymbol{u}^T[\boldsymbol{a}(\boldsymbol{x},\boldsymbol{u})+\boldsymbol{v}]\\&=V_{\boldsymbol{x}}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x},0)+V_{\boldsymbol{x}}(\boldsymbol{x})G(\boldsymbol{x},\boldsymbol{u})\boldsymbol{u}+\boldsymbol{u}^T\boldsymbol{a}(\boldsymbol{x},\boldsymbol{u})+\boldsymbol{u}^T\boldsymbol{v}\\&=V_{\boldsymbol{x}}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x},0)+\boldsymbol{u}^T(G^T(\boldsymbol{x},\boldsymbol{u})V_{\boldsymbol{x}}^T(\boldsymbol{x}+\boldsymbol{a}(\boldsymbol{x},\boldsymbol{u})+\boldsymbol{v})\end{aligned}\tag{3.11}$$

If we introduce the transformation

$$\boldsymbol{\nu}=\boldsymbol{v}+\boldsymbol{a}(\boldsymbol{x},\boldsymbol{u})+G^T(\boldsymbol{x},\boldsymbol{u})V_{\boldsymbol{x}}^T(\boldsymbol{x})\tag{3.12}$$

then (3.11) becomes

$$\dot{S}(\boldsymbol{x},\boldsymbol{u})=V_{\boldsymbol{x}}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x},0)+\boldsymbol{u}^T\boldsymbol{\nu}\tag{3.13}$$

By using transformation (3.12), the cascade system (3.9) becomes

$$\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x},0)+G(\boldsymbol{x},\boldsymbol{u})\boldsymbol{u}\tag{3.14a}$$

$$\dot{\boldsymbol{u}}=-G^T(\boldsymbol{x},\boldsymbol{u})V_{\boldsymbol{x}}^T(\boldsymbol{x})+\boldsymbol{\nu}\tag{3.14b}$$

From Assumption 3.1 and because of $\boldsymbol{u}^T\boldsymbol{\nu}=\boldsymbol{y}^T\boldsymbol{\nu}=\boldsymbol{\nu}^T\boldsymbol{y}$, (3.13) becomes

$$\dot{S}(\boldsymbol{x},\boldsymbol{u})\le\boldsymbol{\nu}^T\boldsymbol{y}\tag{3.15}$$

This means that the cascade system (3.14) is passive with the output $\boldsymbol{y}=\boldsymbol{u}$ and control $\boldsymbol{\nu}$ according to the Definition 3.1.

For system (3.14), let the control $\boldsymbol{\nu}(t)=0$ and the output $\boldsymbol{y}(t)=\boldsymbol{u}(t)=0$ for all t, then $\dot{u}=0$. From (3.14), we can get

$$\dot{x}=f(x,0)\tag{3.16a}$$

$$G^T(\boldsymbol{x},\boldsymbol{u})|_{\boldsymbol{u}=0}V_{\boldsymbol{x}}^T(\boldsymbol{x})=0\tag{3.16b}$$

Then according to Assumption 3.2, we get $\lim_{t\to\infty}\boldsymbol{x}(t)=0$. Together with $u(t)=y(t)=0$, we know that the cascade system (3.14) is zero-state detectable.

Finally, according to the Theorem 3.1, $\boldsymbol{\nu}=-K\boldsymbol{y}=-K\boldsymbol{u},K>0$ achieves globally asymptotical stability of the equilibrium $(\boldsymbol{x},\boldsymbol{u})=(\boldsymbol{x}_s,\boldsymbol{u}_s)$ of system (3.14). Or equivalently to say, the equilibrium $(\boldsymbol{x},\boldsymbol{u})=(\boldsymbol{x}_s,\boldsymbol{u}_s)$ of system (3.9) is globally asymptotically stable if we take $\boldsymbol{\nu}=-K\boldsymbol{u},K>0$ in the transformation (3.12). That is

$$\begin{aligned}\boldsymbol{v}=&-\boldsymbol{K}\boldsymbol{u}+\mathcal{L}\left[\boldsymbol{f}_{\boldsymbol{u}}^T(\boldsymbol{x},\boldsymbol{u})W_{\boldsymbol{x}}^T(\boldsymbol{x})+R\boldsymbol{u}\right]\\&-G^T(\boldsymbol{x},\boldsymbol{u})V_{\boldsymbol{x}}^T(\boldsymbol{x})\end{aligned}$$

□

It should be noted that the feedback control (3.10) includes the cancellation of useful nonlinearities $\boldsymbol{a}(\boldsymbol{x},\boldsymbol{u})$. This means that for general nonlinear systems, by using an arbitrarily Lyapunov-like function $W(\boldsymbol{x})$ rather than a Lyapunov function $V(\boldsymbol{x})$, in order to guarantee the asymptotical stability of systems, an additional control vector $\boldsymbol{v}$ must be introduced into the dynamic controller (3.1b).

The role of Theorem 3.2 is that it provides a tool to analyze the stability behavior of the dynamic controller (3.1b) by way of analyzing the amount of $\boldsymbol{v}$. If $\boldsymbol{v}$ is not zero, the proposed dynamic controller can not guarantee the globally asymptotical stability of the systems. But if the amount of $\boldsymbol{v}$ is small, we still can expect that the system (3.1) should have a larger asymptotically stable region. The ideal case is $\boldsymbol{v}=0$ that means, the dynamic controller can guarantee the globally asymptotical stability without any artificial control vector $\boldsymbol{v}$. From following corollary, we see that it is just the case for affine nonlinear system.

Corollary 3.1. *Consider a nonlinear system which is affine to* $\boldsymbol{u}$, *that is*

$$\dot{\boldsymbol{x}}=\boldsymbol{f}_1(\boldsymbol{x})+G(\boldsymbol{x})\boldsymbol{u}\tag{3.17}$$

and suppose

1. *the equilibrium* $\boldsymbol{x}=0$ *for unforced part* $\dot{\boldsymbol{x}}=\boldsymbol{f}_1(\boldsymbol{x})$ *is globally stable and a* C^2 *positive definite proper function* $V(\boldsymbol{x})$ *is known such that* $V_{\boldsymbol{x}}(\boldsymbol{x})\boldsymbol{f}_1(\boldsymbol{x})\le 0$.
2. *the descent function* $W(\boldsymbol{x})$ *is taken as* $\mathcal{L}^{-1}V(\boldsymbol{x})$, *that is* $\mathcal{L}W(\boldsymbol{x})=V(\boldsymbol{x})$.
3. $\dot{\boldsymbol{x}}=\boldsymbol{f}_1(\boldsymbol{x}),G^T(\boldsymbol{x})V_{\boldsymbol{x}}^T(\boldsymbol{x})=0\Rightarrow\lim_{t\to\infty}\boldsymbol{x}(t)=0$

Then for the cascade system

$$\dot{\boldsymbol{x}}=\boldsymbol{f}_1(\boldsymbol{x})+G(\boldsymbol{x})\boldsymbol{u}\tag{3.18a}$$

$$\dot{\boldsymbol{u}}=-\mathcal{L}\left[G^T(\boldsymbol{x})W_{\boldsymbol{x}}^T(\boldsymbol{x})+R\boldsymbol{u}\right]+\boldsymbol{v},\quad\mathcal{L}>0\tag{3.18b}$$

the equilibrium $(\boldsymbol{x}_s,\boldsymbol{u}_s)$ *can be made globally asymptotically stable without any artificial control* $\boldsymbol{v}$, *that is* $\boldsymbol{v}=0$.

Proof. For the affine nonlinear system (3.17), we have

$$\begin{aligned}\boldsymbol{a}(\boldsymbol{x},\boldsymbol{u}) &= -\mathcal{L}\left[\boldsymbol{f}_{\boldsymbol{u}}^T(\boldsymbol{x},\boldsymbol{u})W_{\boldsymbol{x}}^T(\boldsymbol{x}) + R\boldsymbol{u}\right] \\ &= -\mathcal{L}G^T(\boldsymbol{x})W_{\boldsymbol{x}}^T(\boldsymbol{x}) - \mathcal{L}R\boldsymbol{u}\end{aligned}$$

and $G(\boldsymbol{x},\boldsymbol{u})$ is simply $G(\boldsymbol{x})$. Therefore by Theorem 3.2, the artificial control $\boldsymbol{v}$ becomes

$$\begin{aligned}\boldsymbol{v} &= \boldsymbol{\nu} - \boldsymbol{a}(\boldsymbol{x},\boldsymbol{u}) - G^T(\boldsymbol{x},\boldsymbol{u})V_{\boldsymbol{x}}^T(\boldsymbol{x}) \\ &= \boldsymbol{\nu} + \mathcal{L}G^T(\boldsymbol{x})W_{\boldsymbol{x}}^T(\boldsymbol{x}) - G^T(\boldsymbol{x})V_{\boldsymbol{x}}^T(\boldsymbol{x}) + \mathcal{L}R\boldsymbol{u} \\ &= \boldsymbol{\nu} + G^T(\boldsymbol{x})[\mathcal{L}W_{\boldsymbol{x}}^T(\boldsymbol{x}) - V_{\boldsymbol{x}}^T(\boldsymbol{x})] + \mathcal{L}R\boldsymbol{u} \\ &= \boldsymbol{\nu} + \mathcal{L}R\boldsymbol{u}\end{aligned} \tag{3.19}$$

The condition 3 in this corollary means that the system (3.18) is zero-state detectable. According to Theorem 3.2, take $\boldsymbol{y} = \boldsymbol{u}$ and by using $\boldsymbol{\nu} = -K\boldsymbol{y} = -K\boldsymbol{u}$, the cascade system (3.18) can be made globally asymptotically stable. In such case, $\boldsymbol{v} = (-K + \mathcal{L}R)\boldsymbol{u}$. Take $K = \mathcal{L}R$, then we get $\boldsymbol{v} = 0$. □

We can even make some extension for Corollary 3.1. Actually, if $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})$ can be written in the form

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) = \sum_{i=0}^{k} G_i(\boldsymbol{x})\boldsymbol{u}^i \tag{3.20}$$

and if the unforced part $\dot{\boldsymbol{x}} = G_0(\boldsymbol{x})$ is stable and the Lyapunov function $V(\boldsymbol{x})$ can be found, our dynamic controller can always make the system stable, or asymptotically stable under the assumption of zero-state detectability, as long as we choose the descent function $W(\boldsymbol{x}) = \mathcal{L}V(\boldsymbol{x})$ where $\mathcal{L}$ is a positive constant. In such case, $\boldsymbol{v}$ can always be made zero.

Remark 3.3. In this corollary, we assume that $\mathcal{L}W(x) = V(x)$, that is, the descent function $W(x)$ is in the same level with Lyapunov function $V(x)$ of unforced part. Note that actually, the Lyapunov function $V(x)$ is always a meaningful choice for the descent function $W(x)$ if it is known. The restriction of this assumption is that the $V(x)$ is difficult to be constructed for many nonlinear systems. But until now, most papers on asymptotical controller for nonlinear systems assume the pre-knowledge on Lyapunov function or control Lyapunov function [2],[1], [7].

Remark 3.4. We should make some notes on the factorization (3.7). Our method is based on the factorization of $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})$ as (3.7). Combining (3.7) with the dynamic controller, we construct the system as a cascade, as in (3.2). Here we call $\boldsymbol{f}(\boldsymbol{x},0)$ an *unforced part* of the system and assume the unforced system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},0)$ is stable (not necessary asymptotically stable). Such kind of factorization has been used in some papers such as in [7],[3]. Actually from the identity

$$\begin{aligned}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{x},0) &= \left(\int_0^1 \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x},\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}}\right|_{\boldsymbol{\alpha}=\theta\boldsymbol{u}} d\theta\right)\boldsymbol{u} \\ &\triangleq G(\boldsymbol{x},\boldsymbol{u})\boldsymbol{u}\end{aligned} \tag{3.21}$$

we see that $G(\boldsymbol{x},\boldsymbol{u})$ is a smooth map if $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})$ is smooth.

In paper [1], the authors construct the cascade system by normalizing the original affine nonlinear system based on the relative degree of the system. In such a way, the system is divided into two parts, one represents *zero dynamics* which reflects the internal properties of the system. It is proved that for the *affine nonlinear system*, it can be made passive *if and only if* the zero dynamics are stable. Compared with the Assumption 3.1 in this paper, the assumption of zero dynamics being stable is weaker for affine nonlinear systems that can be made passive through feedback control. But, unfortunately, this is valid only for affine nonlinear systems and usually, the normal form of a nonlinear system is not easy to obtain.

Remark 3.5. In general, the control (3.10) results in cancellation of useful nonlinearities. But From corollary 3.1, we see that for some special nonlinear systems, and if the Lyapunov function of the unforced part is known, such cancellation can be avoided. In other words, the dynamic controller can guarantee the globally asymptotical stability without any compensation. But actually, the original idea to propose our simple dynamic controller is to use it for general nonlinear systems for which we do not have more prior knowledge of Lyapunov function. For its simple structure, the cost is that we can not guarantee the asymptotical stability in general. In such case, for a given problem, we can construct the dynamic controller and determine the asymptotically stable region by some methods like Zubov's successive approximation, which has been used in [4].

4 Simulation

Example 4.1 ([7]). Consider a single input nonlinear system

$$\dot{x}_1 = -x_1^3 + x_1 e^{x_2} u^2 \tag{4.1}$$

$$\dot{x}_2 = x_2^2 u \tag{4.2}$$

Take $W(\boldsymbol{x}) = \frac{1}{2}[q_1 x_1^2 + q_2 x_2^2]$, where q_1 and q_2 are positive, and we can easily calculate

$$\boldsymbol{f}_{\boldsymbol{u}}^T(\boldsymbol{x},\boldsymbol{u}) = [2x_1 e^{x_2} u \quad x_2^2], \quad W_{\boldsymbol{x}}^T(\boldsymbol{x}) = [q_1 x_1 \quad q_2 x_2]^T$$

and therefore

$$\dot{u} = -\mathcal{L}(2q_1 x_1^2 e^{x_2} u + q_2 x_2^3 + Ru) \tag{4.3}$$

Take $\mathcal{L} = 1, q_1 = q_2 = 1, R = 0.1$, the simulation result is shown in Figure 1 with initial condition $(x_1(0), x_2(0), u(0)) = (1, 1, 0)$. Next we analyze the stability property of this controller. Obviously, the Lyapunov function for the unforced part can be taken as $V(\boldsymbol{x}) = W(\boldsymbol{x}) = \frac{1}{2}[q_1 x_1^2 + q_2 x_2^2]$, and $G^T(\boldsymbol{x}, u) = [x_1 e^{x_2} u \quad x_2^2]$. Therefore the stabilizing controller based on Theorem 3.2 can be calculated as

$$\begin{aligned} \dot{u} &= -ku - G^T(\boldsymbol{x}, u) V_{\boldsymbol{x}}^T(\boldsymbol{x}) \\ &= -ku - q_1 x_1^2 e^{x_2} u - q_2 x_2^3 \end{aligned} \tag{4.4}$$

and the aitificial control v becomes

$$v = -ku - a(\boldsymbol{x}, u) - G^T(\boldsymbol{x}, u) V_{\boldsymbol{x}}^T(\boldsymbol{x}) \tag{4.5}$$

$$= (2\mathcal{L}q_1 - 1) x_1^2 e^{x_2} u + (\mathcal{L}R - k) u + (\mathcal{L}q_2 - 1) x_2^3 \tag{4.6}$$

If we choose $\mathcal{L}$ and k properly for given q_1, q_2 and R, v can be made zero. In other words, the proposed dynamic controller can guarantee asymptotical stability under the condition of zero-state detectability without any compensation action v. For this example, we can take $k = 0.1$. The zero-state detectable condition can be verified as follows. With $\nu = 0$ and $y = u = 0$, first according to (4.1), $\lim_{t \to \infty} x_1(t) = 0$. According to (4.4), $-q_2 x_2^3 = 0$, then $x_2 = 0$. By Definition 3.2, we see that the zero-state detectable condition is satisfied.

5 Conclusions

We have derived a kind of dynamic controller called the quickest descent controller for general nonlinear systems based on the idea of decreasing a descent function $W(\boldsymbol{x})$ at each moment. The structure of such dynamic controller is very simple and can be implemented online. We have showed that such a simple controller can consider both the stability and the performance requirements for the system. Many examples showed the effectiveness of such a controller.

By using cascade passivity based theory, we analyzed asymptotical stability of the proposed dynamic controller. We showed that for some special kinds of nonlinear systems such as affine nonlinear systems, our dynamic controller can guarantee the stability of the system as long as the Lyapunov function for the unforced part is known. While for general nonlinear systems, the additional control vector $\boldsymbol{v}$ must be introduced as a compensation factor so that asymptotical stability can be obtained. It should be noted that for general nonlinear systems, without using prior knowledge on the Lyapunov function, the proposed dynamic controller can still be constructed. At that time, the asymptotically stable region can be determined by using methods like Zubov's successive approximation.

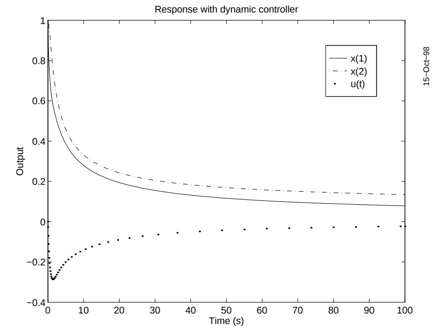

Figure 1: Dynamic controller for Example1

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