On the K-geodeticity of a graph

R.M. RAMOS (1), M.T. RAMOS(2) & J. SICILIA(2)
(1) Departamento de Estadística e Investigación Operativa
Universidad Complutense de Madrid
Avenida Puerta Hierro s/n, Madrid
ESPAÑA

(2) Departamento de Estadística, Investigación Operativa y Computación
Universidad de La Laguna
Avenida Francisco Sánchez s/n, La Laguna, Tenerife
ESPAÑA

Abstract: The study of connectivity properties in graphs and digraphs is of special interest to the designers of reliable communication or interconnection networks. For the network designer in particular it is useful to have some knowledge about those graphs that have high, or maximum, vertex connectivity. Thus, different types of graphs have attracted much interest in recent years. They are characterized for conditions determined in their configurations. A special class of these graphs are those named geodetic graphs. A graph G = (V,E) is said to be geodetic graph, if between any pair of non adjacent vertices x, y $\in$ V there is an unique path. This class of graphs has been studied by several authors. They have obtained some interesting properties of these graphs together with a number of results that connect with other types of graphs.

In this paper we present generalizations of geodetic graphs, which allow three, four, five or , in general, k shortest paths between any two non adjacent vertices. These graphs are called trigeodetic, quatergeodetic,…. or, in general, k-geodetic.

We present some properties about k-geodetic graphs. An upper bound for the number of edges of k-geodetic graphs is obtained. An algorithm to obtain the k-geodeticity of a graph is shown. If we require a k-geodetic graph in which two vertices exist which have exactly k paths of minimum length between them, the graph is said own k-geodetic graph. We obtain some properties about the own k-geodetic graphs.

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1 Introduction

A connected graph is defined by means of the existence of a path between an arbitrary pair of vertices; a forest can be defined on the basis of uniqueness of the paths. Both conditions are fulfilled in the case of trees. Similarly, as we can consider various forms of connectedness, we can consider distinct weaker forms of uniqueness of paths, for example:

(i) Two arbitrary vertices with distance ≤ 2 are connected by at most one shortest path. A graph with this property will be called weakly geodetic.

(ii) Two arbitrary vertices are connected by at most one shortest path. These graphs are called geodetic [4].

(iii) Two arbitrary vertices are connected by at most one path of length less than or equal to the diameter of the graph. We shall call the graphs with this property strongly geodetic .

The concept of geodetic graph is a natural generalization of a tree. A tree is a connected graph whose number of edges is n-1. While in a tree there is a unique path joining any two vertices; in a geodetic graph, there is a unique shortest path connecting any two vertices.

Srinivasan, Opatrny and Alagar [13] in 1988 introduced a new type of graphs, called bigeodetic graphs, which are a generalization of geodetic graphs. Bigeodetic graphs are defined as graphs in which each pair of vertices has at most two paths of minimum length between them.

Geodetic and bigeodetic graphs have been studied in the last years because of their relevance to the design of some interconnection or communication computer networks (see [2], [4], [6], [7], [8] and [14]).

Ramos, Siclia, and Ramos [12] in 1998 introduced a new class of graphs: k-geodetic graphs.

These graphs generalize to the bigeodetic and geodetic graphs. K-geodetic graphs are defined as graphs in which each pair of vertices has at most k shortest paths between them. In this paper we studied some new properties about these class of k-geodetic graphs. Besides we define the own k-geodetic graphs as those k-geodetic graphs which there exist two vertices with exactly k shortest paths between them.

The remaining of this section is devoted to recall some basic concepts and results used throughout this paper. Let $G = (V,E)$ be an undirected simple graph, that is without loops or multiple edges, with set of vertices V and set of edges E. Its adjacency matrix is denoted by $A = (a_{ij})$ $i = 1,2,\ldots,n$; $j = 1,2,\ldots,m$ and is given by $a_{ij} = 1$ if the edge (x_i,x_j) exists in G and $a_{ij} = 0$ if the (x_i,x_j) does not exist in G. For any edge $(x,y)\in E$, we say that x is its initial vertex, and y its final vertex. For any pair of vertices $x, y \in V$, a path $x, x_1,x_2,\ldots,x_{n-1}, y$ from x to y, with its vertices different except possibly x and y, is called an x-y path. We consider only simple connected graphs with at least two vertices. . A tree is a connected graph whose number of edges is n-1. Two vertices x, y are adjacent if the edge (x,y) exists. The distance between any pair of vertices x, y of the graph is the minimum length between both vertices and it is denoted by d(x,y). The diameter of G is the maximum of distances d(x,y) between any vertices x, y of the graph.

We will define a relation on the edges set E of a graph G, as follows: two edges are related by R if they are contained in one and the same circuit of G. The relation R is obviously a equivalence relation. Let $\{E_1, E_2,\ldots, E_n\}$ be the partition of the set E of edges of G induced by R. Let R_i denote the subgraph of G with its edges being the elements of E_i and its vertices the endpoints of the elements of E_i . Each isolated vertex of G forms a further graph R_i. The graphs R_i are called the blocks of G. A vertex p of a graph G is a cutvertex if there are two edges in G incident with p and no circuit in G containing both edges. A non separable graph is a connected graph which has not cutvertices. A block of a graph is a maximal non separable subgraph. If G is non separable, then G itself is often called a block.

Let us examine a path F of a connected graph G with its endvertices p and q belonging to distinct blocks of G and let us record those inner vertices of F which are incident to edges of different blocks. Let order of these vertices along F from p to q be $a_1, a_2,\ldots,a_n$. The sequence $p, R_1, a_1,R_2, a_2,\ldots,R_n, a_n,R_{n+1},q$ is called a block-chain corresponding to F. All blocks in a block-chain are distinct. This uniqueness of block-chain indicates that the blocks of the graph also has a tree like structure.

The Block-Graph G_B of a graph G is made as follows: The vertices $x^{Bi}\in V(G_B)$ are the blocks B_i of the graph G, that is, for each block B_i of G, it will have a vertex x^{Bi} of G_B , and exists an edge (x^{Bi}, x^{Bj}) if $B_i\cap B_j \neq \emptyset$, that is, if they contain some vertices belonging both of them.

The Block-Graph G_B is a sequence of complete subgraphs joined by edges or vertices.

We will define the block-cutvertex graph G^R as follows: The vertices of G^R are associated with each block and with each cutvertex of G. The edges of G^R are obtained by connecting each vertex associated to a block with all the vertices associated with cutvertices in the corresponding block. The block-cutvertex graph of a connected graph G is a tree.

2 K-geodetic graphs.

As we have above mentioned a graph is said to be geodetic is joining any pair of non adjacent vertices there is a unique path. Geodetic graphs have been studied by several authors [2], [4], [5], [10], [13] who analyzed various of their properties. A natural extension of geodetic graphs would be define a new graph, where each pair of vertices has two paths of minimum length between them. A simple graph with that condition is not possible. This is only possible when the graph is a multigraph of order two and these multigraphs have to be complete. The generalizations of these type of graphs are those named k-geodetic graphs, graphs which allow three, four, five or, in general, k shortest path between any two non adjacent vertices.

Definition: Let G be a simple graph, that is, without loops or multiple edges. We will say G is a k-geodetic graph if each pair of vertices has at most k paths of minimum length between them.

Obviously, if two arbitrary vertices with distance $\leq k+1$ are connected by at most k shortest paths, the graph will be called weakly k-geodetic. If two arbitrary vertices are connected by at most k paths of length less than or equal to the diameter of the graph, the graph will be called strongly k-geodetic.

It is obvious that if a graph is k-geodetic then it will be p-geodetic with $p \geq k$. The inverse is not true. Also, if a graph is k-geodetic then it will be weakly k-geodetic. If a graph is strongly k-geodetic then is k-geodetic. But if a graph is strongly k-geodetic it could not be strongly geodetic.

In Ramos et al. [12] we proved the following properties about the k-geodetic graphs.

Proposition 1: If G is a k-geodetic graph then all its blocks are k-geodetic. The converse is not true.

Theorem 1: A separable graph of diameter two is k-geodetic if, and only if, G has exactly one cutvertex, all its blocks are k-geodetic of diameter two at most and all the vertices of G are adjacent to the cutvertex of G.

Theorem 2: Let G be a separable graph where all its blocks are k-geodetic and satisfy the following property: all the vertices of each block B_i are adjacent to any cutvertex of B_i. Then G is k-geodetic.

If we are going to look for upper and lower bounds for the number of edges of a k-geodetic graph with n vertices, a trivial result is obtained for the lower bound which is the number of edges of a tree. So, the minimum number of edges without violating the connectivity will be n-1 and, as any tree is geodetic, then the graph will be k-geodetic. Also an upper bound corresponds to the complete graph which is the combinatorial number $C_{n,2}$. That does not mean that a k-geodetic graph can have any number of edges m. So, a k-geodetic graph with a determined number of edges is not always possible. For example, a graph which has four vertices and five edges cannot be geodetic. The following results proved in Ramos et al. [12] given the way to design connected k-geodetic graphs.

Theorem 3: Given $k \geq 2$ and $n \geq k + d + 1$, it is possible to design a connected k-geodetic graph with n vertices and diameter d, in such a way that the number of edges is

$$d - 2 + k + \binom{k}{2} + k(n-k-d+1) + \binom{n-k-d+1}{2}$$

Proposition 2: Given k, all the connected graphs of diameter two with $n \leq k+2$ vertices are k-geodetic graphs. Besides, this number of vertices is maximal, i.e. there exist at least a connected graph of diameter two with n = k + 3 vertices which is not k-geodetic.

3 Determining the k-geodeticity of a graph.

In this section we propose new results about k-geodetic graphs. An algorithm to obtain the k-geodeticity is presented. The algorithm determines the maximum order of the geodeticity of the graph; that is, the procedure obtains the maximum k value which is allow us join two vertices with exactly k shortest path between them.

If the diameter of a graph is arbitrary, the following result allow us the maximum k-geodeticity of a graph with n vertices.

Proposition 3: The maximum k-geodeticity of a graph with n vertices only depend of the number of vertices and it is $k = 2^{2-s} \times 3^{(n-6+2s)/3}$ if n = s (mod. 3) where $s \in \{0,1,2\}$ and the diameter is d = (n – 6 + 2s) / 3 + 3 – s if n = s (mod. 3).

Proposition 4: Let G be a graph with n vertices and diameter d, then the maximum k-geodeticity is
$k = [(n-2)/d-1)]^{(d-(p+1))} \times [(n-2)/(d-1) + 1]^{p}$ if n-2 = p (mod. (d-1)), p = 0,1,2,…,d-2.

Proposition 5: Let G = (V,E) be a simple graph with diameter d. Let $D = (d_{ij})_{n \times n}$ be the distance matrix of G. Then, it is possible to determine the k-geodeticity of the graph G whose value depend of the above matrices.

Proof: consider the adjacency matrix raised to the r-th power, $A^r = (t_{ij})_{nxn}$ being t_{ij} the number of paths of cardinality r between x_i and x_j (x_i , $x_j \in V(G)$). Let P_r denoted the set $P_r=\{(i,j) / d_{ij} = r\}$ and let k_r be the value $k_r= \max_{(i,j)\in Pr}\{t_{ij}\}$. Then, it is obvious that graph G is k-geodetic, being $k = \max_{1\leq r\leq d} k_r$.

The above proposition allows us to give a procedure to see the k-geodeticity of the graph, that is, the order of the geodeticity of the graph. We will use the following matrices:

- $A(I,J)_{N \times N}$ the adjacency matrix.
- $D(I,J)_{N \times N}$ the distance matrix.
- $T_R(I,J)_{N \times N} = A^R(I,J)$ the R-th power of A.
- $P_R(I,J)_{N \times N}$ a matrix whose elements are 1, is $d_{ij} = R$ and 0 if $d_{ij} \neq R$.

$K(R)_{1\times d}$ a vector of possible values of the k-geodeticity.

ALGORITHM 1:

Step 1:
Compute the adjacency matrix A(I,J).
Compute the distance matrix D(I,J).
Set R = 1.

Step 2:
Compute $T_R(I,J) = A^R(I,J)$ the R-th power of A.
Define an auxiliary matrix $P_R(I,J)$: if $d_{ij} = R$,
then $P_R(I,J) = 1$; otherwise $P_R(I,J) = 0$.

Step 3:
If $P_R(I,J) = 1$, then compute the maximum of the elements of $T_R(I,J)$. Let K(R) be this maximum. Introduce K (R) into a list L.
Set R = R + 1.

Step 4:
If R = diameter d, then compute the maximum value of the list L. Let k be this maximum. Stop.
Otherwise go to step 2.

The k-geodeticity of graph G is the k value obtained in the algorithm above.

We show an example for determining the k-geodeticity of a graph. Consider the graph shown in fig. 1. The diameter of the graph on fig. 1 is d = 3. We compute the distance and adjacency matrices**.**

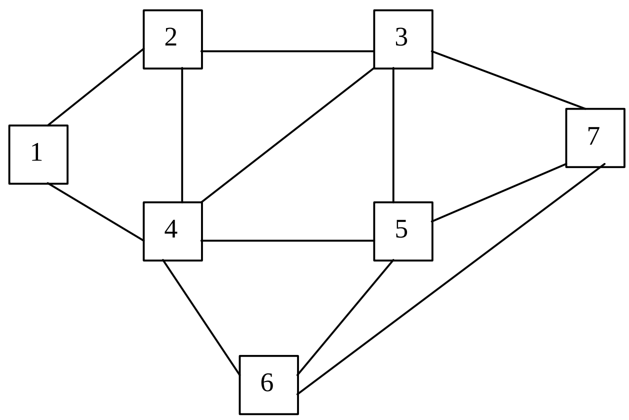

Fig. 1

Set R = 1 and calculate T_1 (I,J), we consider the auxiliary matrix P_1(I,J) and have the value K(1) = 1. We continue the algorithm and we calculate for R = 2 the matrices T_2 (I,J) and P_2 (I,J). So we obtain K(2) = 3, this value is included in the list L. Next we obtain K(3) = 4 and set R = 4, going to the step 4. The diameter is less than or equal R, so we stop. The geodeticity of the graph is 4.

4 Own k-geodetic graphs.

In this section we introduce the new class of own k-geodetic graphs.

Definition: A graph G is named own k-geodetic graph if G is a k-geodetic graph in which two vertices exist which have exactly k paths of minimum length between them.

Logically if we denoted C the class of geodetic graphs, we will see that G can be partitioned in k sub class C_i which will be formed by the own i-geodetic graphs. If a graph is an own k-geodetic graph then we can make up a (k-1)-geodetic graph. It is possible to split one edge to remove one path. A own k-geodetic graph is not own (k-1)-geodetic graph neither viceversa.

A result which relate the k-geodeticity of a graph with the k-geodeticity of its blocks is given next.

Theorem 4: Let G be splitted in r blocks B_1 B_2, ... ,B_r such that any block B_i is own p_i-geodetic ($p_i \le k$) and there exist exactly p_i shortest paths between z_{i1} and z_{i2} where z_{i1}, z_{i2} are any cutvertices of B_i. Then G is k-geodetic with

$$k \le \prod_{i=1}^{r} p_i$$

Besides G will be own k-geodetic graph with

$$k = \prod_{i=1}^{r} p_i$$

if the graph G is designed as a sequence of blocks B_1 B_2, ... ,B_r such that the Block-Graph G_B is a chain, x^{B1}, x^{B2},...,x^{Br}.

Proof: Let G_B be the Block-Graph associated with G and we suppose that we give a weight $w_i = p_i$ (geodeticity of the block B_i) to every vertex x^{Bi} of G_B. Then any path P between x^{Bt} and x^{Bs} will have a weight $\prod p_i$ where $x^{Bi} \in$ P. This mean that any pair of vertices x,y of G, the geodeticity of G is at most $\prod p_i$ where $x^{Bi} \in$P. Now we have two posibilities

(i) if x, y $\in B_i$ $1 \le i \le r$, then there exist at most p_i shortest paths between them. Hence k = $p_i \le \prod p_i$ where $x^{Bi} \in$P;

(ii) if x $\in x^{Bt}$ and y $\in x^{Bs}$ will have at most $\prod p_i$ where $x^{Bi} \in$ P paths between them if that paths go through these blocks B_i, using only cutvertices z_i with $x^{Bi} \in$P.

Since the k-geodeticity of the graph G is obtained by multiplying the feasible maximum numbers p_i, then

$$k \le \prod_{i=1}^{r} p_i$$

Besides, if the Block-Graph is a chain x^{B1}, x^{B2},...,x^{Br} then obviously

$$k = \prod_{i=1}^{r} p_i$$

and hence G will be an own k-geodetic graph.

We have seen by the above theorem that the own k-geodeticity of a graph G can be obtained if the graph have a determined design. Is it possible to obtain the own k-geodeticity on any graph?. The answer is affirmative and we will need only to use a simple algorithm and the definition of block-cutvertex graph.

We have seen that the block-cutvertex graph of a connected graph is a tree. Hence the own k-geodeticity of a graph G is obtained by the following algorithm:

ALGORITHM 2:

Step 1:

Construct the block-cutvertex graph G^R relative to G. Associate to any vertex x_i of G^R, a weight w_i, such that $w_i = p_i$ if x_i is a block-vertex and $w_i = 1$ if x_i is a cutvertex

Step 2:

Determine the shortest path P_{xy} which joins any pair of vertices x, y of G^R. (Observe that there exists an unique shortest path between two given vertices, since the block-cutvertex graph is a tree).

Step 3:

For any path (x-y path) $P_{xy} = \{x = s_1, s_2 ..., s_{t1}, y = s_t\}$.

Calculate the cost $c(x,y) = \prod p_i$ where $s_i \in P_{xy}$.

Step 4:

Determine $k = \max \{c(x,y) / x,y \in G^R\}$ where G^R is the block-cutvertex graph. Then the graph G is an own k-geodetic graph.

The above algorithm is efficient because we would need to calculate l(l-1)/2 costs c(x,y) only, where l is the number of blocks-vertices of the block-cutvertex graph.

5 Results

In this paper we propose new results about k-geodetic graphs. The maximum k-geodeticity of a graph is analyzed. We propose a procedure to determine the k-geodeticity of a graph. Also, the own k-geodetic graphs are defined and some results are commented. We give an algorithm to calculate the own k-geodeticity of a graph.

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