Some types of bifurcations in a bioeconomic model

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Abstract: This paper deals with some aspects of the Theory of Dynamical Systems applied to a model of interespecific competition. A qualitative study is presented where the changes in the number and/or the stability of the system´s equilibrium points is analized. The aim of this study is to stablish the different types of bifurcations that can appear in our model.

Key- Words: fixed point, stability, limit cycle, bifurcations

1.- Introduction. Justification of the model.

The model we introduce is based on a dynamical system which was firstly studied both analitically and experimentally by Gause [4] as a model of competition between two species. It consists of the following two differential equations for the population stock levels x and y:

$$\left.\begin{aligned} \dot{x} &= rx\left(1-\frac{x}{K}\right) - axy \\ \dot{y} &= sy\left(1-\frac{s}{L}\right) - bxy \end{aligned}\right\} \quad (1)$$

In these equations r, s, K, L, α and β are positive parameters with the following meaning:

- r and s are the intrinsic growth rates for species x and y

- K and L are the environmental carrying capacities or saturations levels for each species

- α and β are a measure of the interaction between the two species

An external resource is assumed to exist that supports each population in the absence of the other population according to a logistic law (P.F. Verhulst, [1]). Much information about the system can be obtained from a qualitative approach to the system. An analysis of the isoclines shows distinct types of equilibrium. Basically we can find two situations: *Competitive coexistence* and *competitive exclusion*. In the first case there is a stable node Q = (x_0, y_0) with both x_0 and y_0 being positive (See Figure 1a)[1]. In the second case (Figure 1b) Q is a saddle point and two stable equilibria (nodes) exist at (K,0) and (0,L). The competitive outcome depends on the initial population levels, because one of the species is ultimately driven to extinction. Figure 1c shows also competitive exclusion but only one stable equilibrium exists either at (K,0) or at (0,L). One of the species inevitably wins the competition.

Gause´s equations are structurally stable except for certain special cases, like in Figure 1d. A small change in the position of the isoclines can transform this diagram into a diagram of the type shown in Figure 1b.

Though in certain cases the model predicts the complete exclusion of either population x or y, in the natural environment, a population that is completely out- competed by another population may find various refuges where it can continue to survive, at least in a small number.

Now let the population x be subject to harvesting so that equation (1) becomes:

$$\dot{x} = rx\left(1-\frac{x}{K}\right) - a\,xy - qEx \quad (2)$$

where E is the fishing effort and q is a catchability coefficient.

We can extend the model assuming that the fishing effort itself is a dynamic variable[2] that satisfies

$$\dot{E} = kE\left(x-\frac{c}{pq}\right) \quad (3)$$

where c, p, k are the fishing costs, the catch price and a proportionality coeficient respectively.

The quotient $\frac{c}{pq}$ is the zero- rent population level.

Our model is now a three- dimensional dynamical system which seems to be more complex than the two- dimensional system (1):

[1] Figures taken from Clark, W.[2] C. p.194.

[2] Clark W. C.[2], p.322

$$
\left.\begin{array}{lcl}
\dot{x} & = & rx(1-\frac{x}{K}) - axy - qEx \\
\dot{y} & = & sy(1-\frac{y}{L}) - bxy \\
\dot{E} & = & kE(x-\frac{c}{pq})
\end{array}\right\} \quad (4)
$$

space, a supercritical bifurcation occurs when a stable spiral changes into an unstable spiral sorrounded by a small, nearly elliptical limit cycle. The subcritical case is always much more dramatic. After the bifurcation, the trajectories must jump to a distant attractor which could be a fixed point, another limit cycle, the infnity or- in three and higher dimensions a chaotic attractor (which is, for example, the case of Lorenz attractor).

2.- Previous concepts about bifurcations

The qualitative structure of the flow of a dynamical system can change if parameters are modified. In particular, new fixed points can appear while others can disappear or their original stability can change. These qualitative changes in the dynamics are called bifurcations and the parameter values at which they occur are called bifurcation points.

We define some of types of the bifurcations that may appear for n-dimensional systems with $n \geq 2$:

a) **Saddle- node bifurcation**

This is the basic mechanism for the creation and destruction of fixed points. As a parameter of the system increases or decreases, the fixed points approach each other, then collide and they finally disappear. Even after the fixed points have disappeared, they continue to influence the flow attracting the trajectories towards a Aghost zone@.

b) **Transcritical bifurcation**

This type of bifurcation is held when as a parameter varies, fixed points approach each other, collide and then instead of disappearing they swap their stability.

c) **Pitchfork bifurcation**

In this type of bifurcation, fixed points tend to appear and disappear in symmetrical pairs.

d) **Hopf Bifurcation**

In this type of bifurcation a lost of stability occurs because complex eigenvalues cross the imaginary axis from left to right. There are two types of Hopf bifurcations: *supercritical* and *subcritical*. In terms of the flow in phase

3.- Calculus of fixed points.

Fixed points of system (4) are the solutions of equation f(x) = 0, i.e.

$$
\left.\begin{array}{lcl}
0 & = & rx(1-\frac{x}{K}) - axy - qEx \\
0 & = & sy(1-\frac{y}{L}) - bxy \\
0 & = & kE(x-\frac{c}{pq})
\end{array}\right\} \quad (5)
$$

The system has six fixed points:

P1= (0,0,0), P2= (K,0,0), P3= (0,L,0),

$$P4 = \left(\frac{Ks(-r+a L)}{-rs+b KLa}, \frac{-Lr(s-b K)}{-rs+b KLa}, 0 \right),$$

$$P5 = \left(\frac{c}{pq}, 0, \frac{r(qKp-c)}{q^2 Kp} \right)$$

$$P6 = \left(\frac{c}{pq}, -L\frac{(-psq+b c)}{spq}, -\frac{-srKqp+rsc+a LKsqp-a LKb c}{q^2 Ksp} \right)$$

The Jacobian matrix of the vector field f is:

$$J=\begin{pmatrix} r-\frac{2r}{K}-ay-qE & -ax & -qx \\ -by & s-\frac{2s}{L}y-bx & 0 \\ kE & 0 & k(x-\frac{c}{pq}) \end{pmatrix}$$

4.- Some estimated parameters.

We have considered the real case of a fishery that exploits population x (chup mackerel, *scomber japonicus*[3]) which is competing with population y (anchovy, *engraulis encrasicolus*[4]) for the use of the same resource.

We have taken as a reference the estimates presented for these two species in García Del Hoyo (1997). Therefore we have assigned the following values to the parameters of the system:

K = 10000 Tons	L = 1000 Tons
c = 15000 ptas/day	p = 80000 ptas/Ton
r = 0.75	s = 0.35
k= 0.05	

Our paper deals with the study of the dynamics of the system as function of parameters α, β y q.

5.- Stability of the fixed points P1, P2, P3 and P4.

Stability analysis of points P2, P3 and P4 is easy if we note that for them the fishing effort is zero, so that the dynamics is similar to that of the bidimensional system (1). Equilibrium can be reached at points P2, P3 or P4 in the way shown in Figure 1.

Dynamics near point P1 = (0,0,0) is not outstanding though it can be shown that this is a saddle point because associated eigenvalues are:

$\lambda_1 = r > 0$

$\lambda_2 = s > 0$

$$\lambda_3 = \frac{-kc}{pq} < 0$$

Thus, the origin is an unstable node in two directions and a stable node in one direction.

6.- Stability of point P5

Calculating the charpoly of the Jacobian matrix (6) of the system (4) for point P5, we obtain the associated eigenvalues as:

$$l_1 = \frac{bc}{pq}+s$$

$$l_2 = -\frac{cr+\sqrt{cr(4ckK-4k\,Kpq+cr^2}}{2Kpq}$$

$$l_3 = \frac{-cr+\sqrt{cr(4ckK-4k\,Kpq+cr^2}}{2Kpq}$$

We are interested in studying the cases in which these eigenvalues are negative real numbers or complex numbers with negative real part, so that P5 is an attractive point.

It can be shown that $l_1 < 0 \Leftrightarrow q < \frac{b\,c}{sp} = q_3$

If $q < \frac{4ckK+cr}{4kK^2p} = q_2$ then λ_2, λ_3 □ ▫ and $\lambda_2 < 0$.

In that case, $\lambda_3 < 0 \Leftrightarrow q > \frac{c}{Kp} = q_1$.

Note that $q_1 < q_2$ always[5]. Hence, we obtain the next classification for the associated eigenvalues:

	$q < q_1$	$q_1 < q$ ✁ q_2
$q > q_3$	$\lambda_1 > 0, \lambda_2 < 0, \lambda_3 > 0$	$\lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0$
$q < q_3$	$\lambda_1 < 0, \lambda_2 < 0, \lambda_3 > 0$	$\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$ **P5 is a stable node**

Table 1a: Stability of point P5

[3] Houttuyn, 1782

[4] Linnaeus, 1758

[5] For the assigned values to parameters r, s, K, L, c, p y k, the values of q_1 and q_2 are: $q_1 = 1.875$✎10^{-5} and $q_2 = 1.875703125$✎10^{-5}. The values of q_3 depends on β.

$q > q_2$
$\lambda_1 >0$, Re(λ_2)<0, Re(λ_3)<0
$\lambda_1 <0$, Re(λ_2)<0, Re(λ_3)<0 **P5 is a stable spiral**

Table 1b: Stability of point P5

Remak: If $q = q_1$, then $\lambda_3 = 0$ and linearization about P5 would not predict the nature of this fixed point. The same will apply for λ_1 if $q = q_3$. In both cases it would be necessary the study of the system restricted to the center manifold.

If $q = q_2$, then $\lambda_2 = \lambda_3$ are negative real numbers.

In Figure 2 we have represented a trajectory starting from a point near P5 for values $\alpha = 9.75 \cdot 10^{-4}$, $\beta = 1.62 \cdot 10^{4}$ and $q = 1.8756 \cdot 10^{-5} \in (q_1,q_2)$. In this case P5 = (9996.80, 0, 12.7918) is stable. Trajectory approachs rapidily to P5. Note that this point is close to P2. We will see later that these two points collide for values of q next to that in Figure 2.

Figure 3 shows the same for $q = 7 \cdot 10^{-5}$. P5 is a stable spiral in two directions and a stable node in one direction. In this case, trajectory spirals towards P5 = (2678.57, 0, 7844.38) .

7.- Stability analysis for point P6.

The qualitative study of stability for point P6 as a function of the parameters q, α and β becomes difficult due to the expressions shown by the associated eigenvalues as functions of these parameters. Because of the above reasons we have made use of a numerical study of some particular cases. Taking as a reference a preceding paper[6] in which point P6 was studied for some values of q, α and β, we have fixed $\alpha = 9.75 \cdot 10^{-4}$, and $\beta = 1.62 \cdot 10^{-4}$ and we have made a numerical analysis[7] for values of q from 0 to 1 with step 10^{-9}. Table 2 is a summary of the obtained results. The value q_3 is the same as in Table 1(q_3 =0.0000867857 for the assigned value of b) and q_4 = 0.0002980886049 and q_5 = 0.00032.

[6] Vílchez L.,M.L., Velasco M., F., García Del Hoyo J.J.[11].

[7] Programming with Mathematica

	$0 < q < q_3$	$q_3 < q < q_4$
Eigenvalues sign	Re(λ_1) = Re(λ_2) < 0, λ_3 > 0	Re(λ_1) = Re(λ_2) < 0, λ_3 < 0
Nature of point P6	**Saddle point**	**Stable spiral**

Table 2a: Stability of point P6

$q_4 < q < q_5$	$q_5 < q < 1$
Re(λ_1) =Re(λ_2)>0, λ_3 > 0	λ_1<0 λ_2<0 λ_3 >0
Unstable spiral	**Unstable Node**

Table 2b: Stability of point P6

In Figure 4 we have represented a trajectory starting from an initial point next to P6 for $q = 9 \cdot 10^{-5} \in (q_3,q_4)$. In that case P6 is a stable spiral in two directions and a stable node in one direction. The trajectory spirals towards P6 = (2083.33, 35.71, 6210.31).

8.- Two transcritical bifurcations.

In the light of results in Table 1 and 2, we come to the conclusion that there are two transcritical bifurcations, one between points P2 and P5, and another between P5 and P6.

A) <u>Transcritical bifurcation between points P2 and P5.</u>

If we notice that the associated eigenvalues for point P2 = (K,0,0) are:

$$\lambda_1 = -r < 0$$
$$\lambda_2 = s - \beta K$$
$$\lambda_3 = k\left(K - \frac{c}{pq} \right)$$

it can be shown that P2 is a stable node if $b > \frac{s}{K}$ and $q < \frac{c}{pK} = q_1$, and a saddle point otherwise.

If $b = \frac{s}{K}$ or $q = q_1$ then one of the eigenvalues

λ_1 or λ_2 would be zero. Therefore, linearization would not predict the nature of the critical point and it would be necessary the study of the system restricted to the center manifold.

Comparing these results with those of Table1 for point P5, we find that for $b > \frac{s}{K}$ and $0 < q < q_1$, fixed point P2 is a stable node and P5 is a saddle point. For $q = q_1$ both points collide and for $q_1 < q < q_2$ they exchange their stability and become saddle and node point respectively.

Table 3 is a summary of these results.

If $b > \frac{s}{K}$	$q < q_1$	$q_1 < q < \min(q_2, q_3)$
Eigenvalues sign and nature of point P5	$\lambda_1 <0$, $\lambda_2 <0$, $\lambda_3 <0$ **Stable node**	$\lambda_1 <0$, $\lambda_2 <0$, $\lambda_3 >0$ **Saddle point**
Eigenvalues sign and nature of point P5	$\lambda_1 <0$, $\lambda_2 <0$, $\lambda_3 <0$ **Saddle point**	$\lambda_1 <0$, $\lambda_2 <0$, $\lambda_3 <0$ **Stable node**

Table 3: Transcritical bifurcation between points P2 and P5

Remark: Note that $q_1 < q_2$ always and in the case we are studing ($b > \frac{s}{K}$) it can be shown that $q_1 < q_3$.

B) Transcritical bifurcation between points P5 y P6.

In the light of the results from Tables 1 and 2 we come to the conclusion that, for values of $\alpha = 9.75 \cdot 10^{-4}$ and $\beta = 1.62 \cdot 10^{-4}$, there exists also a transcritical bifurcation between points P5 y P6 for the value $q = q_3 = \frac{b\,c}{sp} = 8.7178 \cdot 10^{-5}$.

Notice that for $q_2 < q < q_3$ the first point is a saddle point and the second one is a stable spiral in two directions and a stable node in one direction. It can be shown that the two points collide for $q = q_3$ and for $q_3 < q < q_4$ they exchange their stability. See Table 4 for a summary.

	$q_2 < q < q_3$	$q_3 < q < q_4$
Eigenvalues sign and nature of point P5	$Re(\lambda_1) = Re(\lambda_2) <0$, $\lambda_3 <0$ **Stable spiral in two directions, stable node in one**	$\lambda_1 >0$, $Re(\lambda_2) = Re(\lambda_3) <0$ **Saddle point**
Eigenvalues sign and nature of point P6	$\lambda_1 >0$, $Re(\lambda_2) = Re(\lambda_3) <0$ **Saddle point**	$Re(\lambda_1) = Re(\lambda_2) <0$, $\lambda_3 <0$ **Stable spiral in two directions, stable node in one**

Table 4: Transcritical bifurcation between points P5 and P6

Figure 5 shows two trajectories starting at points next to P5 and P6 respectively for a value of q greater than the bifurcation value q_3. Notice how the trajectory starting near P5 (unstable in that case) moves away from it and is finally attracted towards P6 (stable spiral).

9.- A Hopf Bifurcation.

If we look at Table 2 we will see that for $q < q_4$, the equilibrium point P6 is a stable spiral in two directions and a stable node in one direction because of the associated eigenvalues are two conjugated complex numbers with negative real parts and a negative real eigenvalue. However, for values of $q > q_4$ situation is reversed and P6 becomes an unstable spiral. The complex eigenvalues cross the imaginary axis and their real parts become positive. This is the first sign that a Hopf bifurcation may exist. In order to confirm this bifurcation, we should find some limit cycle surrounding P6. Numerical integration[8] of the system shows that when P6 becomes unstable, a small attractive limit cycle appears surrounding it. In this case we can confirm that it is a supercritical Hopf bifurcation.

In Figure 7 it can be seen this limit cycle. Figure 8 shows a trajectory starting at a point close to P6 = (629´008, 708´859, 39´205), exactly at point P0 = (629, 708´8, 39´2). As it can be noticed, this trajectory is repelled in spiral from P6 towards the limit cycle. Figure 9 shows a trajectory with initial values at a point close to the

[8] With ODE Software[8]. Figures have been plotted with this software too.

limit cycle but external to it (exactly at P0 = (630, 709, 40). This trajectory is inmediately attracted by the limit cycle.

It is important noting that all the previous properties exist locally but not globally, i.e. they hold in a neighbourhood of the limit cycle and the fixed point.

As we move away from a neighbourhood , the situation can be completely different like the one shown in Figure 10. In this figure it can be seen how a trajectory starting at point P0 = (735, 715, 45) (out of the basin of attraction of the limit cycle) is attracted by one of the other equilibrium points of the system, the point P2 = (0, 1000,0).

We have calculated the period T of the limit cycle, T = $\frac{2p}{w}$, where ω represents the frequency of the limit cycle that can be calculated by the formula[9] ω = Im(λ) where λ is the complex eigenvalue associated to point P6 for the bifurcation value $q = q_4$.

We must stand up that we have made a numerical analysis. Hence the results are not exact and we should realize that there is a small neighbourhood of q_4 such that if q is next to q_4 there will be another limit cycle for q. In Figure11 we have represented some of these limit cycles for values of q in a neighbourhood of q_4 with a radium less than 10^{-7}.

10.- Final remarks.

The preceding study has shown some dinamical properties of system (4) in accordance with parameters values. In the present section we would point out some final remarks:

1) Some parameters, like c or p, are changing constantly and they depend on some economic restrictions. Therefore, if we consider that the catch price does not vary with time, it does not correspond with reality, but we must consider that making a qualitative study with a great number of parameters is difficult and it is necessary to make use of numerical simulations. For simplicity it is convenient to assing fixed values to the parameters. Our next goal is to change the values of the parameters on some intervals and to show all the dynamical possibilities of our system.

2) A problem arises when we try to put the theoretical study into practice. Do results solve our problems in the real enviroment? After analizing the transcritical bifurcation between P2 and P5 we have noticed that for $q<q_4$ an equilibrium point appear at

$$P5=\left(\frac{c}{pq},0,\frac{r(qKp-c)}{q^2Kp}\right)$$

whose third coordinate (which represents the fishing effort) is negative. This is not possible in reality. Something similar occurs between P5 and P6, for the bifurcation value $q=q_3$. The second coordinate of point

$$P6=\left(\frac{c}{pq},-L\frac{(-psq+bc)}{spq},-\frac{-srKqp+rsc+aLKsqp-aLKbc}{q^2Ksp}\right)$$

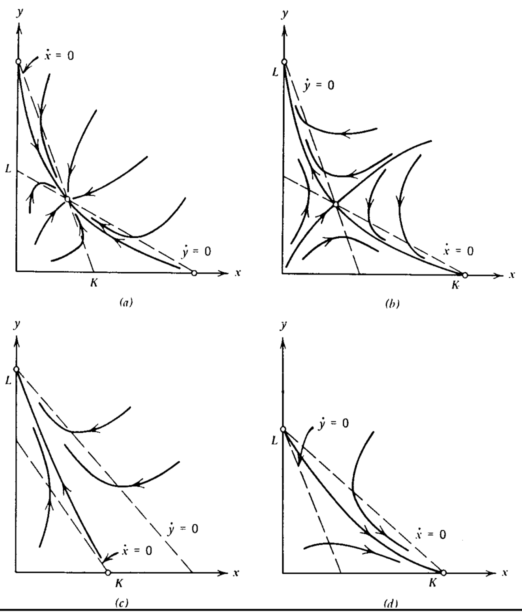
is negative

Let us remark again that this is a theoretical study. Probably for some other models with different values for α and β we would obtain different conclusions.

3) This is a mathematical approach to a biological model. Our next aim is to apply it to a model of optimal management of the resource (by using the Optimal Control Theory). Then it will become a complete bioeconomic model . In such a model we would try to study the optimal exploitation paths.

[9] Strogatz [9],p.51.

11.- Figures.



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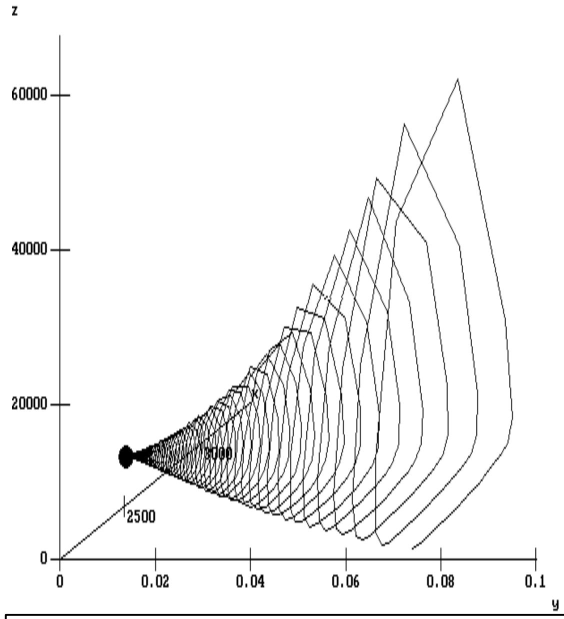
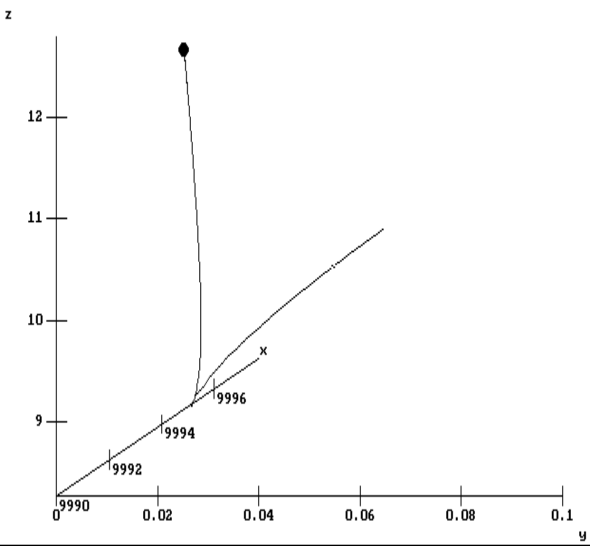


Figure 3



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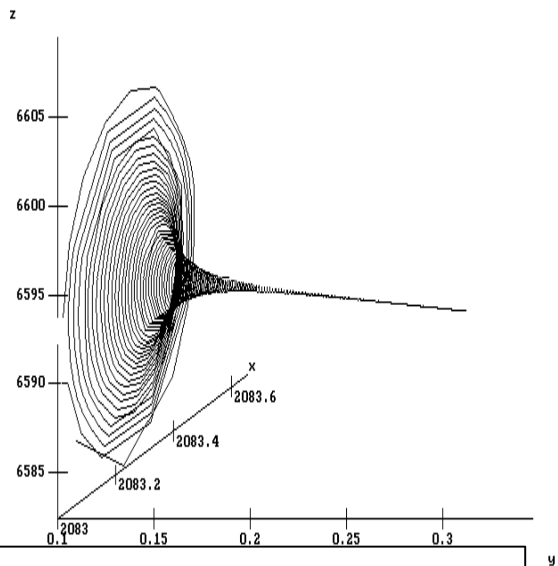

Figure 4

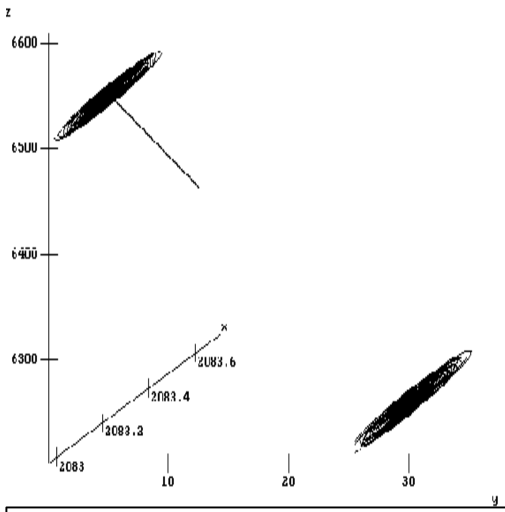

Figure 5

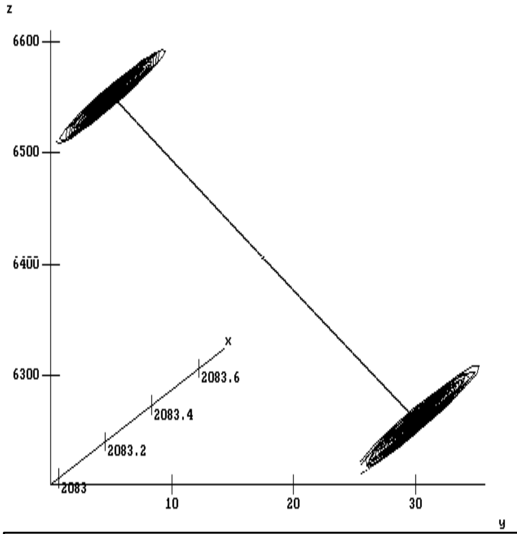

Figure 6

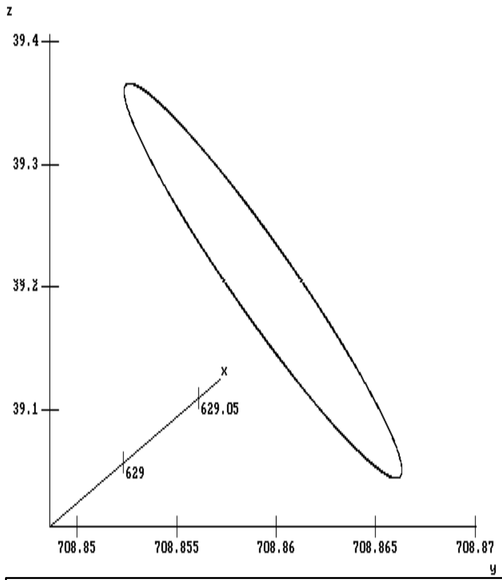

Figure 7

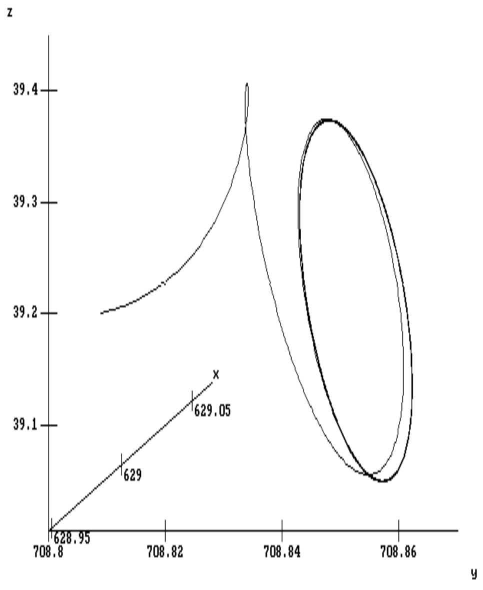

Figure 8

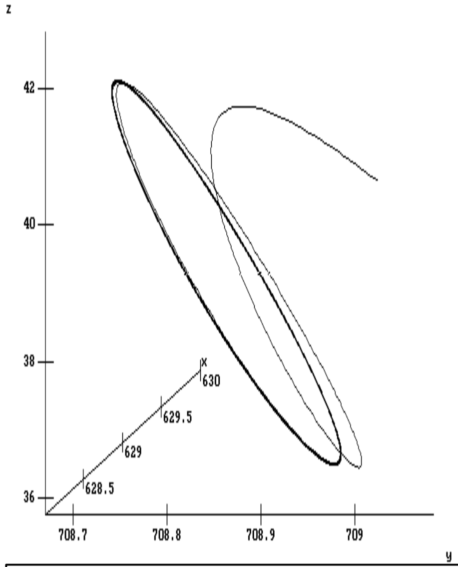

Figure 9

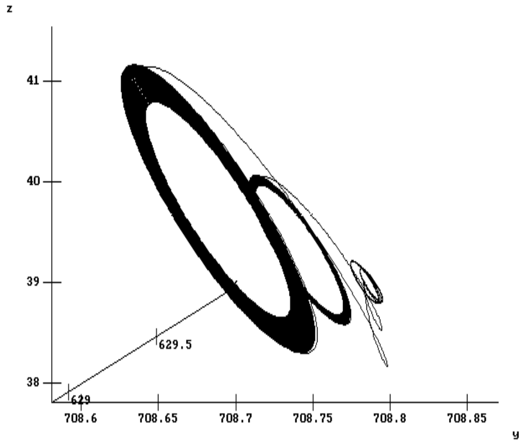

Figure 11

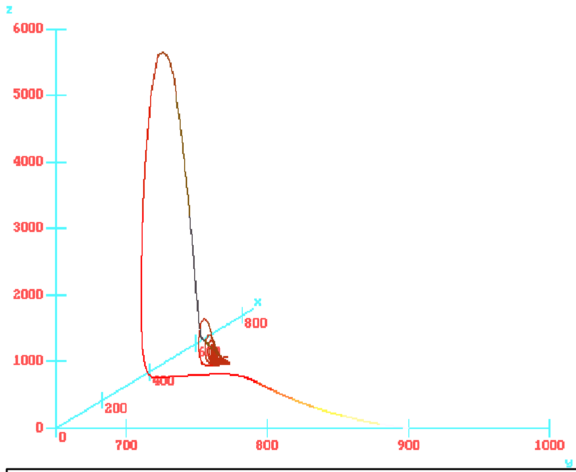

Figure 10

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