Disturbance Decoupling of Singular Systems via P-D Feedback

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Abstract: **-** The problem of disturbance decoupling of singular systems via proportional plus derivative (P-D) feedback is solved. The necessary and sufficient condition for the problem to be solvable is established as a rank condition of the transfer function of the open loop system. A special feedback law solving the disturbance decoupling problem is derived. IMACS/IEEE CSCC'99 Proceedings, Pages:6121-6123

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1 Introduction

The category of linear time invariant singular systems is described by the equation [1]

$$\tilde{E}\dot{x}(t)=\tilde{A}x(t)+\tilde{B}\tilde{u}(t)+\tilde{D}\tilde{\xi}(t) \quad , \quad \tilde{y}(t)=\tilde{C}x(t) \quad (1.1)$$

where $\tilde{E},\tilde{A}\in\mathbb{R}^{n\times n}$, x is the state vector, u is the $\tilde{m}\times 1$ input vector, ξ is the $\tilde{\zeta}\times 1$ unknown disturbance vector and $\tilde{y}$ is the performance output vector. The system is assumed to be regular i.e. $\det\left(s\tilde{E}-\tilde{A}\right)\neq 0$. The rank of the matrix $\tilde{E}$ can be equal to n or less than n. Equation (1.1) describes systems modelled by differential and algebraic equations (f.e. large scale industrial systems, constrained manipulators).

The first results for the disturbance decoupling of singular systems were derived in [2] using a proportional plus derivative feedback law of special type. This P-D controller was of special type, namely the proportional feedback matrix was restricted to be the product of an arbitrary constant and the respective derivative feedback matrix. The motivation for such a restriction was the duality between normal systems $\left(\det\tilde{E}\neq 0\right)$ and singular systems facilitating the recasting of the problem to the decoupling problem of normal systems via pure P feedback. In the same time the problem has also been studied via P state feedback (see [3]-[6]).

In this paper the problem of disturbance decoupling of singular systems is studied. The problem is treated without any restrictions upon the P-D feedback law. The necessary and sufficient condition is derived. Furthermore a special feedback law solving the problem is determined.

It is important to mention that the respective results for the case of input-output decoupling have been derived in [7, 8].

2 Problem Formulation

To system (1.1) apply the P-D state feedback law

$$\tilde{u}(t)=\tilde{F}_D\dot{x}(t)+\tilde{F}_P x(t)+\tilde{\omega}(t) \qquad (2.1)$$

where $\tilde{\omega}$ is the $\tilde{m}\times 1$ vector of external commands. The design goal is to find a feedback law of the form (2.1) such that the output vector is not affected by the disturbances. The definition of the problem can formally be expressed as follows:

Definition 2.1: The problem of *disturbance rejection,* via P-D state feedback, is solvable for system (1.1) if there exist feedback matrices $\tilde{F}_D$, $\tilde{F}_P$ which satisfy the equation

$$\tilde{C}\left(s\tilde{E}-\tilde{A}-s\tilde{B}\tilde{F}_D-\tilde{B}\tilde{F}_P\right)^{-1}\left[\tilde{B} \mid \tilde{D}\right]=\left[\tilde{H}(s) \mid 0\right]; \quad \tilde{H}(s)\in[\mathbb{R}(s)]^{\tilde{p}\times\tilde{m}} \tag{2.2}$$

where $\mathbb{R}(s)$ is the field of rational functions. □

According to Definition 2.1 the closed-loop system must be regular, i.e. $\det(sE-A-s\tilde{B}\tilde{F}_D-\tilde{B}\tilde{F}_P)\not\equiv 0$.

3 Reformulation of the Problem

To reformulate the problem in a form elegant to be solved, it is necessary to distinguish the input terms that they are dependent upon the disturbances and so they do not influence the observability matrix of the closed loop system. To this end, let $\tilde{J}_D$ be an $\tilde{\zeta}\times\tilde{\zeta}$ column rearrangement matrix ($\tilde{J}_D^{-1}=\tilde{J}_D^{\mathsf{T}}$) having the property of distinguishing the independent columns of $\tilde{D}$, i.e. $\tilde{D}\tilde{J}_D=\left[\tilde{D}_1 \mid \tilde{D}_2\right]$ where $\tilde{D}_1\in\mathbb{R}^{n\times\zeta}$ and $\operatorname{rank}\left[\tilde{D}_1 \mid \tilde{D}_2\right]=\operatorname{rank}\tilde{D}_1=\zeta$. Furthermore, consider the $\tilde{\zeta}\times\tilde{\zeta}$ column rearrangement matrix $\tilde{J}_B(\tilde{J}_B^{-1}=\tilde{J}_B^{\mathsf{T}})$ having the property of distinguishing the columns of $\tilde{B}$ being independent among themselves and the columns of $\tilde{D}$ ($\tilde{B}\tilde{J}_B=\left[\tilde{B}_1 \mid \tilde{B}_2\right]$ where $\tilde{B}_1\in\mathbb{R}^{n\times m}$ and $\operatorname{rank}\left[\tilde{B}_1 \mid \tilde{B}_2 \mid \tilde{D}\right]=\operatorname{rank}\tilde{B}_1+\operatorname{rank}\tilde{D}=m+\zeta$). Define the invertible matrix B_D, having the property

$$B_D\left[\tilde{B} \mid \tilde{D}\right]\begin{bmatrix} J_B & 0\\ 0 & J_D\end{bmatrix}= =B_D\left[\tilde{B}_1 \mid \tilde{B}_2 \mid \tilde{D}_1 \mid \tilde{D}_2\right]= =\left[B \mid DB_2 \mid D \mid DD_2\right]$$

where $B^{\mathsf{T}}D=0$ and where B_1 and B_2 are appropriate dependence matrices. Clearly, a matrix B_D having the above properties can always be constructed.Using B_D, the system (1.1) can be rewritten as follows:

$$E\dot{x}(t)=Ax(t)+Bu(t)+D\xi(t)+DB_2\bar{u}(t)+DD_2\bar{\xi}(t) \quad , \quad \tilde{y}(t)=\tilde{C}x(t) \tag{3.1}$$

where $E=B_D\tilde{E},\ A=B_D\tilde{A}$ and

$$u(t)=\left[I_m \quad 0\right]J_B^{\mathsf{T}}\tilde{u}(t) \quad ,\bar{u}(t)=\left[0 \quad I_{\tilde{m}-m}\right]J_B^{\mathsf{T}}\tilde{u}(t),$$
$$\xi(t)=\left[I_\zeta \quad 0\right]J_D^{\mathsf{T}}\tilde{\xi}(t)\ ,\ \bar{\xi}(t)=\left[0 \quad I_{\tilde{\zeta}-\zeta}\right]J_D^{\mathsf{T}}\tilde{\xi}(t)$$

According to the above rearrangement of the elements of the input and the disturbance vector, the feedback law (2.1) can also be rewritten as follows:

$$u(t)=F_D\dot{x}(t)+F_Px(t)+\omega(t); F_D=\left[I_m \quad 0\right]J_B^{\mathsf{T}}\tilde{F}_D\ ,$$
$$F_P=\left[I_m \quad 0\right]J_B^{\mathsf{T}}\tilde{F}_P\ ,\omega=\left[I_m \quad 0\right]\tilde{\omega} \tag{3.2a}$$
$$\bar{u}(t)=\bar{F}_D\dot{x}(t)+\bar{F}_Px(t)+\bar{\omega}(t); \bar{F}_D=\left[0 \quad I_{\tilde{m}-m}\right]J_B^{\mathsf{T}}\tilde{F}_D,$$
$$\bar{F}_P=\left[0 \quad I_{\tilde{m}-m}\right]J_B^{\mathsf{T}}\tilde{F}_P,\ \bar{\omega}=\left[0 \quad I_{\tilde{m}-m}\right]\tilde{\omega} \tag{3.2b}$$

The above partitioning is of significant importance to the disturbance decoupling problem.

Proposition 3.1: The solvability of the disturbance decoupling problem as well the transfer function from the disturbances and inputs to the outputs of the closed loop system with rejected disturbances are independent from the second part of the feedback law, namely the law (3.2b).

Proof: From the definition of the disturbance decoupling and expressions (3.1) and (3.2), it is clear that the problem is solvable if and only if

$$\tilde{C}\left(sE-A-sBF_D-BF_P-sDB_2\bar{F}_D-DB_2\bar{F}_P\right)^{-1}\times \times\left[B \mid DB_2 \mid D\right]=\left[\bar{H}(s) \mid 0\right];\ \bar{H}(s)\in[\mathbb{R}(s)]^{\tilde{p}\times\tilde{m}} \tag{3.3}$$

Since

$$\tilde{C}\left(sE-A-sBF_D-BF_P-sDB_2\bar{F}_D-DB_2\bar{F}_P\right)^{-1}D= =\tilde{C}(sE-A-sBF_D-BF_P)^{-1}D\times \times\left[I_{\tilde{m}-m}-B_2(s\bar{F}_D+\bar{F}_P)(sE-A-sBF_D-BF_P)^{-1}D\right]^{-1}$$

the second set of equations in (3.3) is satisfied if and only if $\tilde{C}(sE-A-sBF_D-BF_P)^{-1}D=0$. Using this equality and since

$$\tilde{C}\left(sE-A-sBF_D-BF_P-sDB_2\bar{F}_D-DB_2\bar{F}_P\right)^{-1}\times \times\left[B \mid DB_2\right]= \tilde{C}(sE-A-sBF_D-BF_P)^{-1}\left[B \mid DB_2\right]\times \left\{\begin{bmatrix} I_m & 0\\ 0 & I_{m-\tilde{m}}\end{bmatrix}-\begin{bmatrix} 0\\ s\bar{F}_D+\bar{F}_P\end{bmatrix}\times \times(sE-A-sBF_D-BF_P)^{-1}\left[B \mid DB_2\right]\right\}^{-1}$$

the first set of eq. in (3.3) holds true if and only if $\tilde{C}(sE-A-sBF_D-BF_P)^{-1}[B \mid DB_2]=[\bar{H}(s) \mid 0]$. ■

From Proposition 3.1, the problem has been reduced to that of finding F_D and F_P, such that

$$\tilde{C}(sE-A-sBF_D-BF_P)^{-1}\left[B \mid D\right]=\left[\bar{H}(s) \mid 0\right] \ ; \quad \bar{H}(s)\in[\mathbb{R}(s)]^{\tilde{p}\times\tilde{m}} \tag{3.4}$$

Eventhough, the second part of the feedback, i.e. the feedback law (3.2b), does not affect the closed loop transfer function (from inputs and disturbances to outputs), it could be useful when studying the stability or pole assignment of the closed loop system, after disturbance decoupling.

According to (3.4) the closed loop system must be regular, i.e. $\det(sE-A-sBF_D-BF_P)\not\equiv 0$. Hence it is necessary to hold that

$$\operatorname{Rank}[B^{\perp}(sE-A)]=n-m \tag{3.5}$$

where $\operatorname{Rank}[\bullet]$denotes the rank of a matrix over the field $\mathbb{R}(s)$, $B^{\perp}$ is a $(n-m)\times n$ full row rank matrix being orthogonal to the full column rank matrix B, namely the left orthogonal of B, ($B^{\perp}B=0$), i.e.

$$\operatorname{rank}\begin{bmatrix} B^{\perp}\\ B^{+}\end{bmatrix}=n; \quad B^{+}=(B^TB)^{-1}B^T \tag{3.6}$$

B^{+} denotes the left inverse of B ($B^{+}B=I_m$).

Let $p=\operatorname{Rank}\left[\tilde{C}(sE-A)^{-1}B\right]$. Also let $\tilde{J}_C$ be a $\tilde{p}\times\tilde{p}$ row rearrangement matrix ($\tilde{J}_C^{-1}=\tilde{J}_C^{\mathsf{T}}$) having the property of distinguishing the independent rows of $\tilde{C}(sE-A)^{-1}B$, i.e. $\tilde{J}_C\tilde{C} = \begin{bmatrix} C \\ \bar{C} \end{bmatrix}$ where $C \in \mathbb{R}^{p\times n}$ and $\text{Rank}\left[C(sE-A)^{-1}B\right] = p$. According to the above definition and the properties

$$\begin{aligned}
&\text{Rank}\left[\tilde{C}(sE-A-sBF_D-BF_P)^{-1}\left[\begin{array}{c|c} B & D \end{array}\right]\right] = \\
&= \text{Rank}\left[\tilde{C}(sE-A-sBF_D-BF_P)^{-1}B\right] = \\
&= \text{Rank}\left[\tilde{C}(sE-A)^{-1}B\right]
\end{aligned}$$

$$\begin{aligned}
&\text{Rank}\left[\tilde{C}(sE-A)^{-1}\left[\begin{array}{c|c} B & D \end{array}\right]\right] = \\
&= \text{Rank}\left[\tilde{C}(sE-A)^{-1}B\right] = \\
&= \text{Rank}\left[C(sE-A)^{-1}B\right] \quad (3.7)
\end{aligned}$$

resulting directly from (3.4), it is concluded that if the condition (3.7) is satisfied then the problem is recasted to that of satisfying the equation

$$C(sE-A-sBF_D-BF_P)^{-1}\left[\begin{array}{c|c} B & D \end{array}\right] = \left[\begin{array}{c|c} H(s) & 0 \end{array}\right] \ ; \quad H(s) \in [\mathbb{R}(s)]^{p\times m} \quad (3.8)$$

4 Problem Solution

Theorem 4.1: The problem of disturbance decoupling is solvable, via P-D feedback, if and only if

$$\begin{aligned}
&\text{Rank}\left[\tilde{C}\left(s\tilde{E}-\tilde{A}\right)^{-1}\left[\begin{array}{c|c} \tilde{B} & \tilde{D} \end{array}\right]\right] = \\
&= \text{Rank}\left[\tilde{C}\left(s\tilde{E}-\tilde{A}\right)^{-1}\tilde{B}J_B\begin{bmatrix} I_m \\ 0 \end{bmatrix}\right]
\end{aligned}$$

Proof: According to the necessary condition (3.7) the problem is reduced to the equation (3.8). Using the definitions of E, A, B, D, condition (3.7) is reduced to the condition of the theorem. To satisfy (3.8) consider the feedback law $u = F_D\dot{x} + F_P x + \omega$ where

$$F_D = B^+E, \quad F_P = -B^+A + \begin{bmatrix} K \\ C \end{bmatrix} \quad (3.9)$$

where $K \in \mathbb{R}^{(m-p)\times n}$ is an arbitrary matrix. Applying this feedback law to equation (3.8) it is readily observed that the condition is satisfied. Furthermore application of this feedback law to the system $(sE-A, B, D, C)$ leads to the closed loop system

$$KX(s) = \Omega_O(s) \quad (3.10a)$$

$$CX(s) = \Omega_R(s) \quad (3.10b)$$

$$(sB^{\perp}E - B^{\perp}A)X(s) = B^{\perp}D\Xi(s) \quad (3.10c)$$

where $X(s) = \pounds\{x(t)\}$, $\Xi(s) = \pounds\{\xi(t)\}$, $\Omega(s) = \pounds\{\omega(t)\}$ $\Omega_O(s) = \Omega(s)\left[I_{m-p} \mid 0\right]^{\top}$, $\Omega_R(s) = \Omega(s)\left[0 \mid I_p\right]^{\top}$ and where the relation $B^+D = 0$ has been used. From (3.5) and since $\text{Rank}[C(sE-A)^{-1}B] = p$, the closed loop system is regular, i.e. $\det(sE-A-sBF_D-BF_P) \not\equiv 0$, via appropriate choice of K. Furthermore, choosing $\bar{F}_D = \bar{F}_P = 0$, the regularity of the original closed loop system ($\det(s\tilde{E}-\tilde{A}-s\tilde{B}\tilde{F}_D-\tilde{B}\tilde{F}_P) \not\equiv 0$) is also guaranteed. ■

Based upon the proof of Theorem 4.1 the following corollary can readily be established.

Corollary 4.1: A special P-D feedback law solving the problem of disturbance decoupling is

$$\tilde{F}_D = \begin{bmatrix} B^+E \\ 0 \end{bmatrix} J_B, \quad \tilde{F}_P = \begin{bmatrix} -B^+A + \begin{bmatrix} K \\ C \end{bmatrix} \\ 0 \end{bmatrix}$$

where K is arbitrary matrix preserving the regularity of the closed loop system, i.e. satisfying the condition $\text{Rank}\begin{bmatrix} K^{\top} & C^{\top} & [B^{\perp}(sE-A)]^{\top} \end{bmatrix}^{\top} = n$. ■

5. Conclusions

The problem of disturbance decoupling of singular systems via P-D feedback has been solved. The necessary and sufficient condition for the problem to be solvable has been established as a rank condition of the transfer function of the open loop system. A special feedback law solving the disturbance decoupling problem has been derived.

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