Wavelets, Squeezed States, and Entropy Problems

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Abstract

It is pointed out that the inhomogeneous Lorentz group forms the mathematical framework for wavelets. The representations of this group applicable to symmetry properties of wavelets are discussed in detail. Since squeezed states are also representations of the Lorentz group, wavelets can be interpreted in terms of the symmetry properties of squeezed states. It is shown that wavelets are space-time squeezed states. This space-time property allows us to examine more carefully the question of whether light waves are photons. Within the present mathematical framework of wavelets, it is possible to define the "window" which allows us to introduce a Lorentz-covariant cut-off procedure. It is possible to make a transition from light waves to photons through this window. The information lost through this process results in an increase in entropy. It is shown that the entropy is not invariant under Lorentz boosts, but the entropy difference remains invariant.

Summary

This is a preliminary summary of the paper I wishes to submit to this conference. My primary research base has been and still is the Lorentz group. Of course, this mathematical language was generated from the study of special relativity and Lorentz transformations of relativistic particles, but it is applicable to many different branches of physics including quantum and classical optics and diagonalization processes for coupled systems. In 1939 [1], Wigner introduced translations to this group resulting in inhomogeneous Lorentz group.

Wigner once remarked that quantum mechanics is the physics of Fourier transformations while special relativity is the physics of Lorentz transformations. The word "wavelet" is relatively new in physics [2], but its concept was formulated in the 1960s [3] in terms of the Lorentz group. The wavelet combines the traditional Fourier transformation with dilation and translational symmetries. In other words, wavelets combine both the physics of Lorentz transformations and the physics of Fourier transformations.

We shall use various aspects of wavelets to discuss a number of long-standing and current topics in physics, including photon localization problem, squeezed states of light, windows specifying limitations in measurement, and increase in entropy due to less-than-perfect measurements. We shall discuss also change in entropy due to various wavelet transformations.

First, we shall point out transformation properties of wavelets can can be derived from those of the Lorentz group. We observe that Lorentz boosts are squeeze transformations which contract and expand two orthogonal directions in two-dimensional space. The squeeze/expansion properties of wavelets can be derived from the squeeze/expansion properties of the Lorentz group. In fact, this is the way in which the concept of wavelets was first developed in the 1960s [3]. The squeezed state of light is also based on this property [4]. The word "squeezed state" is relatively new in physics, but squeeze transformation has been everywhere in physics for many years [5].

The translational symmetry of the inhomogeneous Lorentz group allows us to introduce the concept of "window" which defines a finite interval in which measurements are possible, while it is not possible to measure anything outside this window. Indeed, this is what we do in the real world. Mathematical formulas we use in physics are usually defined for variables extending to infinity from minus infinity. On the other hand, it is not possible to measure all the values in laboratories. Thus, the concept of window is attached to every measurement process. With these theoretical tools, we shall discuss some specific physical problems.

Let us start with the photon localization problem. It is widely believed that the transition from classical theories to quantum counterparts is well understood in all branches of physics, but it is not true for light waves which are classical objects. Their quantum counterparts are photons. Then, are photons light waves? We are not quite ready to say YES to this question. From the traditional theoretical point of view, the answer is NO [6]. However, this negative answer does not prevent us from examining how close photons are to waves by employing the mathematical technique of wavelets and windows. While we cannot solve the photon localization completely, we can gain a better insight by placing wavelets between waves and photons.

Photons are relativistic particles requiring a covariant theoretical description. Classical optics based on the conventional Fourier superposition is not covariant under Lorentz transformations. However, wavelets can be regarded as representations of the Lorentz group [2, 3, 7]. In one of papers which I published with my collab-

orators [8], we discussed the difference between waves and wavelets without mentioning the word "wavelet." We compared there the waves with the covariant harmonic oscillator formalism [9] using the light-cone coordinate system. We have seen there that the lack of covariance of light waves is due to the lack of Lorentz invariance of the integral measure, while the integral measure in the oscillator formalism is invariant. We concluded there that an extra multiplicative factor is needed to make Fourier optics covariant. We point out that this procedure corresponds to the wavelet formalism of wave optics.

In spite of the covariance of wavelets, we shall not be able to prove that photons are wavelets. Instead, we shall make a quantitative analysis of the difference between these two clearly defined physical concepts. The advantage of this quantitative approach is that we can see how close they are to each other. In this way, we can assert that photons are waves with a proper qualification.

In order to establish this practical approach to the photon localization problem, we note first that the internal space-time symmetry of massless particles is governed by Wigner's $E(2)$-like little group [1]. The little group is the maximal subgroup of the Lorentz group whose transformations leave a given four-momentum invariant. In the case of massless particles, the same little group can accommodate particles with different momenta in the same direction. We are thus led to the concept of the extended little group [10], which includes the boost along the direction of the momentum. In this way, we shall show that wavelets are representations of the extended $E(2)$-like little group.

Another convenient feature of the localized wavelet representation is that it is possible to introduce an cut-off procedure in a covariant manner, so as to preserve the information given in the distribution. By introducing the concept of window [11, 12, 13, 14], it is possible to define the region in which the frequency distribution is non-zero. We can then compare the "windowed" wavelet to the photon field in quantum electrodynamics to pinpoint the difference between the photons and wavelets. This concept of window allows us to quantify the difference between the intensity of light wavelets and the photon numbers.

Then a new question arises. How can we quantify the degree of inaccuracy due to this windowing process. One way to deal with this problem is to borrow Feynman's rest of the universe. In his book on statistical mechanics [15], Feynman makes the following statement. *When we solve a quantum-mechanical problem, what we really do is divide the universe into two parts - the system in which we are interested and the rest of the universe. We then usually act as if the system in which we are interested comprised the entire universe. To motivate the use of density matrices, let us see what happens when we include the part of the universe outside the system.* The failure to observe the rest of the universe results in an increase in entropy [16].

It is thus a great challenge to formulate the entropy difference between the windowed distribution and analytical distribution in terms of Feynman's rest of the universe. While this problem is not yet completely understood, we can discuss the covariance property of the entropy change. Without computing entropy, it is possible to prove that, while the entropy is not an invariant quantity, the entropy difference is invariant [17]. I hope to be able to report more concrete results at the conference.

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