Necessary and Sufficient Conditions for Multivariable Pole Placement and Entire Eigenstructure Assignment through Constant Output Feedback

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Abstract : The epoch-making advent of entire eigenstructure assignment (EEA) in late 1970s culminated in potential tools of specifying the time response of linear multivariable systems. The recent pervasive use of EEA stipulates further theoritical investigation. The EEA through constant output feedback is of paramount significance for always but states are not directly available. This paper establishes necessary and sufficient conditions for pole placement and EEA of general linear multivariable systems via constant output feedback.

Keywords : Multivariable pole placement, entire eigenstructure assignment via constant output feedback.

1 Introduction

The multivariable pole placement and entire eigenstructure assignment have been intensively studied over the last three decades, e.g. [1,2,3]. It is well known that the transient response of a linear multivariable system is a linear combination of the dynamical modes of the system. The shape of a mode is specified by its associated eigenvector, and its time-domain characteristic by its associated eigenvalue. The EEA dictates the time response of the system; in other words, the eigenvalues, their associated eigenvectors, and the reciprocal eigenvectors all contribute to the time response. It has been shown that [1] for systems in which $m+l-1 \geq n$ where $m, l,$ and n are the number of inputs, outputs, and states, respectively, arbitrary pole placement is possible via constant output feedback. In this paper, necessary and sufficient conditions for pole placement and EEA of general linear multivariable systems via constant output feedback are derived. The solvability of these conditions for the aforementioned systems is consequently guaranteed.

2 Conditions Derivation

Consider a linear multivariable system described by the following state and output equations:

$$\dot{X}(t) = AX(t) + Bu(t) \tag{1}$$

$$y(t) = CX(t) \tag{2}$$

where $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{l\times n}$, and B

and C without loss of generality are full rank. A constant matrix $G \in R^{m \times l}$ is sought for to place the poles and/or assign the entire eigenstructure of the closed loop system, i.e. λ_i 's and v_i 's $(i = 1, ..., n)$ as eigenvalues and associated eigenvectors.

Replacing,

$$u(t) = Gy(t) \tag{3}$$

yields in,

$$(A + BGC)v_i = \lambda_i v_i \qquad i = 1, ..., n \tag{4}$$

or equivalently,

$$\begin{pmatrix} -\lambda_i I + A & B \end{pmatrix} \begin{pmatrix} v_i \\ q_i \end{pmatrix} = 0 \qquad i = 1, ..., n \tag{5}$$

in which,

$$q_i = GCv_i \qquad i = 1, ..., n \tag{6}$$

or in compact form,

$$G.CV = Q \tag{7}$$

where $\begin{pmatrix} v_i \\ q_i \end{pmatrix}$'s for $i = 1, ..., n$ belong to the Kernel spaces of matrices $\begin{pmatrix} -\lambda_i I + A & B \end{pmatrix}$, correspondingly.

Patently, necessary and sufficient condition for the last equation to yield G is that all rows of Q be linear combinations of rows of CV (Note that CV stands for $C.V$). If v_i 's are chosen such that V be a full rank matrix (which in case of distinct eigenvalues is necessarily so), then by Silvester's Formula CV is full rank. Consequently, matrix $\begin{pmatrix} CV \\ Q \end{pmatrix} = \begin{pmatrix} Cv_1 & ... & Cv_n \\ q_1 & ... & q_n \end{pmatrix}$ has the same row rank as matrix CV. On account of the equality of row and column ranks of a matrix, the following are concluded:

Case 1: $l \leq n$

Matrix CV has (full) row rank l, hence matrix $\begin{pmatrix} CV \\ Q \end{pmatrix}$ has (full, when $l = n$, and deficient when $l < n$) column rank l; i.e., exactly l vectors of n vectors $\begin{pmatrix} Cv_1 \\ q_1 \end{pmatrix}, ..., \begin{pmatrix} Cv_n \\ q_n \end{pmatrix}$ are linearly independent. Putting the argument in reverse direction, the last statement means that the row rank of matrix $\begin{pmatrix} CV \\ Q \end{pmatrix}$ is l. Regarding that the row rank of CV is l, this is tantamount to rows of Q being linear combinations of rows of CV.

In brief, necessary and sufficient conditions for pole placement and EEA of general linear multivarible systems for distinct eigenvalues are that $\begin{pmatrix} v_i \\ q_i \end{pmatrix}$'s $(i = 1, ..., n)$ be chosen such that $V = \begin{pmatrix} v_1 & ... & v_n \end{pmatrix}$ be full rank, and exactly l vectors of n vectors $\begin{pmatrix} Cv_1 \\ q_1 \end{pmatrix}, ..., \begin{pmatrix} Cv_n \\ q_n \end{pmatrix}$ be linearly independent.

Case 2: $l > n$

Matrix CV has (deficient) row rank n (and full column rank n), hence matrix $\begin{pmatrix} CV \\ Q \end{pmatrix}$ has (full) column rank n; i.e., all of the n vectors $\begin{pmatrix} Cv_1 \\ q_1 \end{pmatrix}, ..., \begin{pmatrix} Cv_n \\ q_n \end{pmatrix}$ are linearly independent. Putting the argument in reverse direction, the last statement means that the row rank of matrix $\begin{pmatrix} CV \\ Q \end{pmatrix}$ is n. Regarding that the row rank of CV is n, this is equivalent to rows of Q being linear combinations of rows of CV.

In brief, necessary and sufficient conditions for pole placement and EEA of general linear multivarible systems for distinct eigenvalues are that $\begin{pmatrix} v_i \\ q_i \end{pmatrix}$'s $(i = 1, ..., n)$ be chosen such that V be full rank, and all of the n vectors $\begin{pmatrix} Cv_1 \\ q_1 \end{pmatrix}, ..., \begin{pmatrix} Cv_n \\ q_n \end{pmatrix}$ be linearly independent.

Note that such systems are among the class of systems in which $m + l - 1 \geq n$.

The above argument is summarized in the following theorem.

Theorem : For the linear multivariable system described by the following state and output equations:

$$\dot{X}(t) = AX(t) + Bu(t)$$

$$y(t) = CX(t)$$

where $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{l\times n}$, and B and C without loss of generality are full rank, necessary and sufficient conditions for multivariable pole placement and/or entire eigenstucture assignment of the closed loop system, i.e. λ_i 's and v_i 's $(i = 1, ..., n)$ as distinct eigenvalues and associated eigenvectors, through constant output feedback are that $\begin{pmatrix} v_i \\ q_i \end{pmatrix}$'s (i=1,...,n), satisfying equations (1) through (7), be chosen such that V be full rank and exactly l' vectors of n vectors $\begin{pmatrix} Cv_1 \\ q_1 \end{pmatrix}, ..., \begin{pmatrix} Cv_n \\ q_n \end{pmatrix}$ be linearly independent, where $l' = l$ when $l \le n$ and $l' = n$ when $l > n$.

Corollary 1 : When $l \le n$ evidently there exists a full rank matrix M such that matrix $T = \begin{pmatrix} C \\ M \end{pmatrix}$ be nonsingular. Applying the state transformation $Z = TX$, system equations are transformed to,

$$\dot{Z}(t) = TAT^{-1}Z(t) + TBu(t) \qquad (8)$$

$$y(t) = [I_{l\times l} \quad 0]Z(t) \qquad (9)$$

Hence, regarding that the above similarity transformation does not alter the eigenvalues and eigenvectors of the closed loop system, the second condition of the Theorem is equivalent to " exactly l' vectors of n vectors $(\, v_{1,1} \;\; ... \;\; v_{1,l} \;\; q_1^T \,)^T, ...,$ $(\, v_{n,1} \;\; ... \;\; v_{n,l} \;\; q_n^T \,)^T$ be linearly independent ".

Corollary 2 : When $l > n$ the second condition is trivial. Because, if v_i 's are chosen linearly independent, then CV has full column rank, consequently all of the vectors $\begin{pmatrix} Cv_1 \\ q_1 \end{pmatrix}, ..., \begin{pmatrix} Cv_n \\ q_n \end{pmatrix}$ are linearly independent.

Remarks :

A. Each vector $\begin{pmatrix} v_i \\ q_i \end{pmatrix} \in R^{(n+m)\times 1}$ for $i = 1, ..., n$ is spanned by m vectors $S_{i,1}, ..., S_{i,m}$, being the basis vectors of the null spaces of matrices $(\, -\lambda_i I + A \quad B\,)$, correspondingly.

B. The number of eigenvectors (or similarly $\begin{pmatrix} v_i \\ q_i \end{pmatrix}$ vectors) to be assigned is n, yielding that there are $m \times n$ unknowns to be delved into.

C. A straightforward but tedious procedure to follow to find eigenvectors satisfying the derived conditions is as follows:

C.1. Form :
$\begin{pmatrix} v_i \\ q_i \end{pmatrix} = x_{i,1}S_{i,1} + ... + x_{i,m}S_{i,m}$
for $i = 1, ..., n$

C.2. Solve for $m \times n$ unknowns $x_{i,j}$,
$i = 1, .., n$, $j = 1, ..., m$, satisfying,

C.2.1. V be full rank (and C.2.2 if $l < n$)

C.2.2. All $(l+1)\times(l+1)$ submatrices of m matrices $\begin{pmatrix} v_{1,1} & \cdots & v_{n,1} \\ \vdots & \vdots & \vdots \\ v_{1,l} & \cdots & v_{n,l} \\ q_{1,j} & \cdots & q_{n,j} \end{pmatrix}$, $j = 1, ..., m$ which are $(l+1) \times n$, be singular.

D. The required output feedback matrix is given by:
$G = Q.CV^T.(CV.CV^T)^{-1}$ when $l \le n$ (and preferably $G = Q.CV^{-1}$ when $l = n$)
and when $l > n$ an infinite number of solutions which can easily be found as follows:
Solve directly for $m \times l$ entries of G from $m \times n < m \times l$ equations: $G.CV = Q$; $m \times (l-n)$ entries can be selected arbitrarily.
A class of solutions are obtained as follows:
Find n independent rows of CV to form an $n \times n$ nonsingular submatrix CV' of CV,
Form G' consisting of the corresponding columns of G,
Find G' from $G' = Q.CV'^{-1}$,
Evolve G by juxtaposing columns of G' and zero columns in place of other (left) columns of G; i.e., the corresponding outputs are not fed back.
If other (left) columns of G are wanted to be nonzero columns, then in the above formula Q must be substituted with $Q - G''.CV$ where

G'' is formed by juxtaposing zero columns instead of columns of G' and other (left) nonzero columns of G.

E. The multivariable pole placement and/or entire eigenstructure assignment via constant output feedback can be solved by duality.

When λ_i 's and w_i 's $(i = 1, ..., n)$ are distinct eigenvalues and reciprocal eigenvectors of the closed loop system, i.e.,

$$w_j^T v_i = \delta_{ij} \qquad i, j = 1, ..., n \tag{10}$$

and,

$$w_i^T (A + BGC) = \lambda_i w_i^T \tag{11}$$
$$i = 1, ..., n$$

equivalently it can be written as,

$$\begin{pmatrix} -\lambda_i I + A^T & C^T \end{pmatrix} \begin{pmatrix} w_i \\ \xi_i \end{pmatrix} = 0 \tag{12}$$
$$i = 1, ..., n$$

in which,

$$\xi_i = G^T B^T w_i \qquad i = 1, ..., n \tag{13}$$

or in compact form,

$$G^T . B^T W = \Xi \tag{14}$$

where $\begin{pmatrix} w_i \\ \xi_i \end{pmatrix}$'s for $i = 1, ..., n$ belong to the null spaces of matrices $\begin{pmatrix} -\lambda_i I + A^T & C^T \end{pmatrix}$, correspondingly.

Similarly, necessary and sufficient condition for the last equation to yield G^T is that all rows of Ξ be linear combinations of rows of $B^T W$. Therefore, by similar argument, necessary and sufficient conditions of the above theorem would change to: "$\begin{pmatrix} w_i \\ \xi_i \end{pmatrix}$'s $(i = 1, ..., n)$, satisfying equations (10) through (14), be chosen such that $W = \begin{pmatrix} \xi_1 & \dots & \xi_n \end{pmatrix}$ be full rank and exactly m' vectors of n vectors $\begin{pmatrix} B^T w_1 \\ \xi_1 \end{pmatrix}, ..., \begin{pmatrix} B^T w_n \\ \xi_n \end{pmatrix}$ be linearly independent, where $m' = m$ when $m \leq n$ and $m' = n$ when $m > n$", followed by corresponding corollaries and remarks.

F. In case of repeated eigenvalues and deficient rank matrices B and C, Theorem's conditions change to: "$\begin{pmatrix} v_i \\ q_i \end{pmatrix}$'s $(i = 1, ..., n)$ be chosen such that rank of V equals the number of independent eigenvectors of $A + BGC$ (which must be checked after finding G) and rank of $\begin{pmatrix} CV \\ Q \end{pmatrix} = l'$, where l'=rank of CV when rank of $CV \leq n$ and $l' = n$ otherwise", followed by corresponding corollaries and remarks.

3 Illustrative Examples

The following examples demonstrate the procedure of finding eigenvectors satisfying the Theorem's conditions.

Example 1 : For the linear multivariable system $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ -0.5 & 0.5 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \\ 0.5 & 0.5 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ carry out pole placement and/or entire eigenstructure assignment for the desired closed loop eigenvalue set $\Lambda = \{-1, -2, -5\}$.

Applying the state transformation $Z = TX$ where $T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$ system equations are converted to $A' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B' = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$, $C' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Solving for the basis vectors of the null spaces of matrices $\begin{pmatrix} -\lambda_i I + A & B \end{pmatrix}$, (or those of the transformed system) where $\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -5$, they are found as,

$S_{1,1} = \begin{pmatrix} 1 & 0 & 1 & -1 & 0 \end{pmatrix}^T$,
$S_{1,2} = \begin{pmatrix} 0 & 1 & 0 & -1 & 1 \end{pmatrix}^T$,
$S_{2,1} = \begin{pmatrix} 1 & 0 & 2 & -2 & -2 \end{pmatrix}^T$,
$S_{2,2} = \begin{pmatrix} 0 & 1 & -1 & - & 37 \end{pmatrix}^T$,

$S_{3,1} = \begin{pmatrix} 1 & 0 & 5 & -5 & -20 \end{pmatrix}^T,$
$S_{3,2} = \begin{pmatrix} 0 & 1 & -4 & -1 & 21 \end{pmatrix}^T,$

Solvig,

$$\det\begin{pmatrix} x_{1,1} & x_{2,1} & x_{3,1} \\ x_{1,2} & x_{2,2} & x_{3,2} \\ x_{1,1} & 2x_{2,1}-x_{2,2} & 5_{3,1}-4x_{3,2} \end{pmatrix} \neq 0,$$

$$\det\begin{pmatrix} x_{1,1} & x_{2,1} & x_{3,1} \\ x_{1,2} & x_{2,2} & x_{3,2} \\ -x_{1,1}-x_{1,2} & -2x_{2,1}-x_{2,2} & -5x_{3,1}-x_{3,2} \end{pmatrix} = 0,$$

$$\det\begin{pmatrix} x_{1,1} & x_{2,1} & x_{3,1} \\ x_{1,2} & x_{2,2} & x_{3,2} \\ x_{1,2} & -2x_{2,1}+3x_{2,2} & -20x_{3,1}+21x_{3,2} \end{pmatrix} = 0,$$

the following are found among the infinite number of solutions:
$X_1 = \{x_{1,1} \neq 0, x_{1,2} = 0,$
$x_{2,1} = 6x_{2,2} \neq 0, x_{3,1} = 1.5x_{3,2} \neq 0\},$
$X_2 = \{x_{1,1} = 1, x_{1,2} = -1,$
$x_{2,1} = -3, x_{2,2} = 2, x_{3,1} = 3, x_{3,2} = 1\},$
$X_3 = \{x_{1,1} = 2, x_{1,2} = -1,$
$x_{2,1} = -3, x_{2,2} = \frac{3}{4}, x_{3,1} = 3, x_{3,2} = \frac{3}{2}\},$
$X_4 = \{x_{1,1} = 1, x_{1,2} = -1,$
$x_{2,1} = -3, x_{2,2} = \frac{1}{3}, x_{3,1} = 3, x_{3,2} = \frac{5}{3}\},$
and the corresponding output feedback matrices would be,
$G_1 = \begin{pmatrix} -1 & -7 \\ 0 & -9 \end{pmatrix}$, $G_2 = \begin{pmatrix} -4 & -4 \\ -10 & -9 \end{pmatrix}$,
$G_3 = \begin{pmatrix} -3 & -5 \\ -5 & -9 \end{pmatrix}$, and $G_4 = \begin{pmatrix} -2.5 & -5.5 \\ -\frac{10}{3} & -9 \end{pmatrix}$, respectively.

Example 2 : For the linear multivariable system $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ -0.5 & -0.5 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \\ 0.5 & 0.5 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ carry out pole placement and/or entire eigenstructure assignment for the desired closed loop eigenvalue set $\Lambda = \{-1, -2, -5\}$.
Basis vectors $S_{i,j}$'s are the same as those of Example 1. Solving for 8 unknown entries of matrix G, four of the infinite number of solutions which their fourth and third columns are chosen $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$, are,
$G_1 = \begin{pmatrix} 13 & 14 & -7 & 0 \\ 18 & 18 & -9 & 0 \end{pmatrix}$,
$G_2 = \begin{pmatrix} 15 & 16 & -8 & 1 \\ 22 & 22 & -11 & 2 \end{pmatrix}$,
$G_3 = \begin{pmatrix} -1 & 0 & 0 & -7 \\ 0 & 0 & 0 & -9 \end{pmatrix}$,
$G_4 = \begin{pmatrix} 3 & 4 & -2 & -5 \\ 6 & 6 & -3 & -6 \end{pmatrix}$, respectively.

4 Conclusion

The entire eigenstructure assignment via constant output feedback is a powerful tool of specifying the time response of linear multivariable systems when states are not directly available. The necessary and sufficient conditions for pole placement and EEA via constant output feedback of general linear multivariable systems were derived. As it is observed, multivariable pole placement through constant output feedback involves eigenvector assignment; in other words, the multivariable pole placement and EEA via constant output feedback are in effect identical.

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