Stability Analysis of Multivariable Fuzzy Control Systems Using the FAST Toolbox

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Abstract: - This paper studies the stability analysis of multivariable fuzzy control systems. Particularly, the stability of Takagi-Sugeno (TS) systems is addressed. The results presented in the paper make possible the application of both input-output and frequency response methods to TS systems identified from Input-Output data. The paper also presents the Fuzzy Algorithm Stability Tool (FAST) toolbox developed in the project Fuzzy Algorithm for Control of Multiple-Input Multiple-Output Fuzzy Systems (FAMIMO). Some comparative examples illustrate the application of the proposed techniques. CSCC'99 Proc.pp.7191-7196

Key-Words: - Stability, Fuzzy Control, Fuzzy Models, MIMO systems, SoftwareToolbox, Non-linear systems.

1 Introduction

The stability analysis of multivariable non-linear feedback systems is a complex problem where is difficult to obtain general results [7]. Furthermore, most results on the stability analysis of fuzzy control systems are related to local stability around an equilibrium point [13]. Only some authors have studied the global stability involving all the space in which the variables associated to the process to be controlled can vary [6]. However, the techniques presented in this paper can also be applied to the global stability analysis of multivariable fuzzy control systems. These global problems rise, for instance, when an unstable plant is controlled with a controller that saturates.

This paper summarises several results on the stability of multivariable fuzzy control system obtained in the FAMIMO project. The paper concentrates on the stability of continuous time Takagi-Sugeno (TS) systems. These systems can be easily computed from input-output data of the process to be controlled [11]. Furthermore, rule-based heuristic knowledge can be also included in the model. It should be noted that several stability analysis techniques could not be applied directly to TS models. Particularly, the Input-Output and the Frequency Response techniques are straightforward applied only to linear system with static nonlinearities in the feedback loop [6][9][10]. The work done in FAMIMO extends the above Input/Output and Frequency Response stability techniques to systems with the process modelled by a TS multivariable model and a fuzzy feedback controller as shown in Fig. 1a. The feedback structure in this figure is transformed into the structure shown in Fig. 1b (Lur'e problem), with a linear part and a feedback non-linear part composed by the nonlinearties of the TS model of the process and the fuzzy controller.

Several techniques could also be applied to Mamdani type fuzzy control systems.

Some of the more significant up-to-date stability analysis techniques for MIMO fuzzy systems have been implemented in a Matlab Toolbox called FAST, developed in the FAMIMO project.

The paper is organised as follows: Section 2 summarises the main features of the toolbox. Section 3 presents several comparative examples, showing the possibilities of the different methods. Finally, Section 4 is for the conclusions.

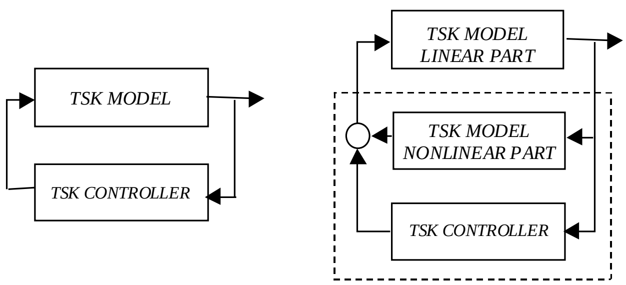

Fig. 1 a) TS fuzzy model with a Fuzzy controller.
b) Linear/Non-linear part decomposition.

2 The FAST Toolbox

A Matlab toolbox called FAST (Fuzzy Algorithm Stability Tool) has been developed. The main features of this tool are shown in Fig. 2. The toolbox implements several methods for stability analysis of MIMO fuzzy systems. Some of those methods have been recently developed in the frame of the FAMIMO ESPRIT Project. In the following a description of the implemented methods is performed.

2.1 State space qualitative analysis of fuzzy control systems

The tool includes traditional heuristic techniques for the stability analysis, such as the linguistic trajectory method and the representation of the trajectories in the phase portrait. Furthermore, stability and robustness indices [1] based on the qualitative theory of non-linear dynamical systems are also being included. These indices have been recently extended in the FAMIMO project for the application to TS fuzzy systems.

2.2 Frequency response methods

The existing frequency response techniques for the analysis of fuzzy control systems have been traditionally applied to systems with a known linear model of the process and a non-linear feedback (fuzzy controller). This model requirement was a significant limitation for practical usage in fuzzy control engineering. However, using the transformation presented above, the stability of TS systems can be studied with these methods.

In FAMIMO the harmonic balance equation has been used to search for limit cycles. In this case this equation leads to

$$G(j\omega)N(a)y = -y \qquad (1)$$

where $y_i = a_i e^{j(\omega t+\theta i)}$ are the complex representation of sinusoids, $G(j\omega)$ is the frequency response of the linear part, and *N(a)* is the describing function of the non-linear part (see Fig 1b). The describing function matrix of the feedback non-linear part is computed by obtaining its response to sinusoidal inputs and calculating the first harmonic of this response.

For a limit cycle to exist (1) should have a nontrivial solution. To solve that equation, in the case where the non-linearity is additively decomposable a method suggested by Mees can be used [6]. For a square *G* this method is based on the fact that (1) can only have a solution if $G^{-1}(j\omega) + N(a)$ has at least one zero eigenvalue. To check whether that happens the Gershgorin theorem is employed [10]. This theorem leads to the stability condition:

$$\left|\hat{g}_{kk}(j\omega) + n_{kk}\right| > \sum_{i\neq k}^{n}\left|\hat{g}_{ik}(j\omega)\right| + \sum_{i\neq k}^{n}\left|n_{ik}\right| \quad k = 1,2,\cdots,n \qquad (2)$$

A graphical tool to analyse the stability using the Gershgorin´s bands has been implemented. The intersection of the Gershgorin circles of the multivariable linear system with the describing function of the non-linear feedback system determines the critical regions to study the stability.

A different method is based on the direct analysis of the harmonic balance equation. Thus, the number of encirclements of the characteristic loci of $G(j\omega) + N(a)$ around the point (*-1,0*) is studied [8].

A third method also developed in FAMIMO is based on the robust analysis of limit cycles using singular values. The method is applied to a system with a multiplicative error Δ model. A theorem to assure the absence of limit cycles has been presented [3]. This theorem defines the following condition on the model error:

$$\overline{\sigma}(\Delta(j\omega)) < \frac{1}{\overline{\sigma}(G(j\omega)N(a)(I + G(j\omega)N(a_0))^{-1}} \quad \forall\omega, \forall a. \qquad (3)$$

This new method for the robust analysis of limit cycles using singular values is also available in the FAST toolbox.

2.3 Input-Output stability

If *G* is a linear time-invariant representation of the process to be controlled, its gain can be easily computed using its frequency response:

$$g(G) = \sup_{\omega} \overline{\sigma}(G(j\omega)) \qquad (4)$$

where σ is the maximum singular value of the matrix $G(j\omega)$. Furthermore, if *H* is a non-linear static fuzzy controller, $H = \phi(e)$, its gain can be obtained as:

$$g(\phi) = \sup_{|e|\neq 0}\left\{\frac{|\phi(e)|}{|e|}\right\} \qquad (5)$$

The small gain theorem states that a sufficient condition for the stability of the closed-loop system in Fig. 1.b is that $g(G)g(H) < 1$. That is, the product of the gains should be lower than 1.

Several stability criteria can be stated from the applications of the above concepts. Particularly, the circle criterion has been applied for the stability analysis of both SISO and MIMO Mandani type fuzzy control systems [9]. The conicity criterion (Multivariable Circle Criterion) can be used to generalise the above results. This criterion leads to the following sufficient conditions for the stability of the fuzzy control system [7]: $i_c < 1$ where i_c is called the conicity index, which is defined as

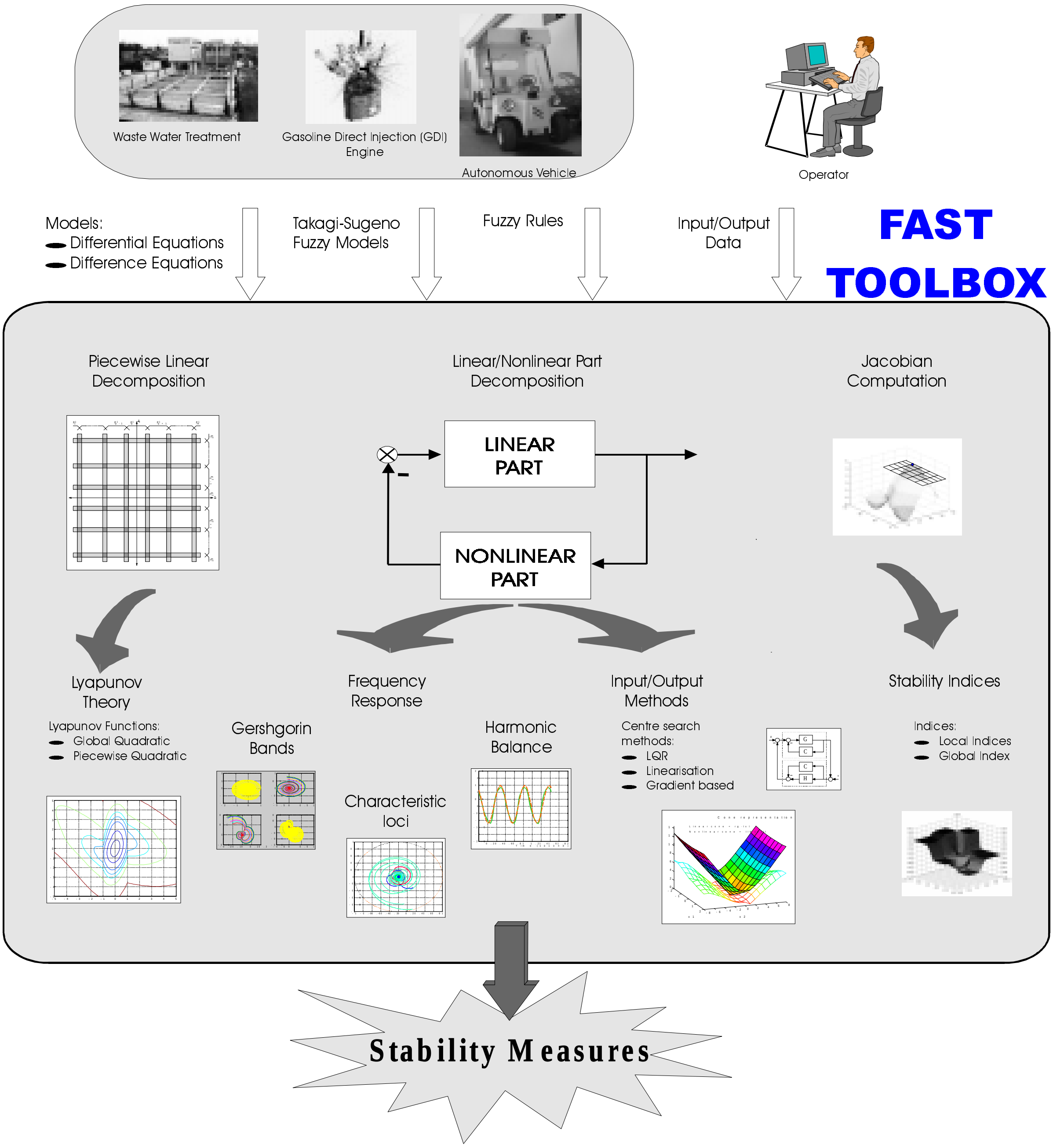

Figure 2. Fuzzy Algorithm Stability Toolbox for Matlab.

$i_c = r_h / r_g$, where $r_h(C) = g(H - C)$ is the conic deviation and $r_g(C) = 1/g(G(I + CG)^{-1})$ is the conic robustness. In the above expressions C is called centre of the cone. Thus, stability analysis is reduced to find a centre such that $i_c < 1$.

Usually, these methods are difficult to apply, and require training to perform block transformation in the control loop and to select parameters. The tool helps to overcome these practical problems. For example, three different methods have been implemented in FAST to obtain a centre: computation of the closest linear system to the non-linear part, optimal linear feedback and optimisation with the gradient descent method.

2.4 Lyapunov analysis using LMI: quadratic stability and piecewise quadratic stability

A continuous time linear TS fuzzy model is composed of rules such as:

$$\text{IF } x \text{ is } L_i \text{ THEN } \dot{x} = A_i x + B_i u;\ i = 1 \cdots M \qquad (6)$$

This system can be considered as a polytopic linear differential inclusion (PLDI). Stability analysis of PLDI and, consequently, of linear TS systems, is reduced to find a common matrix P [11][13], valid for a set of linear matrix inequalities on the form $P > 0$ and $PA_i + A_i^T P < 0$, that can be solved by means of new convex optimization techniques [2]. The quadratic Lyapunov function is given by $V(x) = x^T P x$.

The Lyapunov approach is usually very conservative. Thus, it is not possible to find quadratic Lyapunov functions of many stable fuzzy control systems. However, in case of membership functions with local support, which induce a partition in the space of the input variables, piecewise quadratic Lyapunov functions [5], offer many solutions to stability problems that cannot be solved with conventional techniques. A matrix P_i has to be computed for every region and the Lyapunov function is defined by $V(x) = x^T P_i x$.

These developments came from the work on the stability of heterogeneous control system. Thus, a tool for the implementation of Lyapunov stability analysis of fuzzy multivariable systems using the LMI software provided by another partner of the FAMIMO project [4] has been used. That includes the computation of a matrix to satisfy the quadratic stability of piecewise linear systems (globally quadratic Lyapunov function), or a set of matrices to satisfy piecewise quadratic stability. The FAST tool facilitates the application of the Lyapunov method to TS systems generating automatically the fuzzy partition and obtaining the LMI input data from the fuzzy system definition.

3. Examples

This section presents some examples of the stability analysis of fuzzy systems. Firstly, in order to show the relevance of global stability analysis it will be considered an unstable fuzzy system controlled by a fuzzy controller. When no saturation of the control action exist, the fuzzy controller is able to globally stabilise the system (see Fig 3.a). However, if the control action is saturated, new equilibria and limit cycles could appear, as the saturation level is increased (see Fig 3b-3c). These phenomena could not be detected with conventional techniques as Lyapunov direct method or Conicity criterion.

Example 1: Affine TS system

This example deals with an affine TS system, and has been proposed by Johansson et al. [5] to illustrate the use of piecewise quadratic Lyapunov (PWQL) functions in a two dimensional case (see Fig. 4).

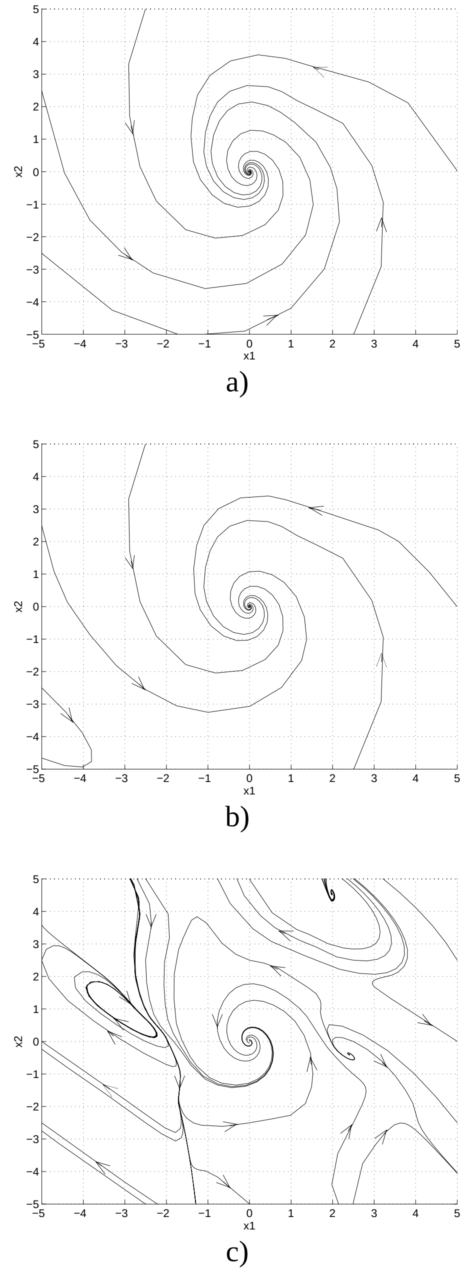

Figure 3. a) System without control action saturation. b) -60 < u < 60. c) -20 < u <20.

The premises of the rules induce a state space partition into 25 regions (9 operating regions and 16 interpolation ones). Using PWQL (see Section 2) it is possible to compute a matrix P_i for every region, that ensures the stability of the global system:

$$P_1 = \begin{bmatrix} 0.0084 & 0.0060 & -0.0002 \\ 0.0060 & 0.0054 & 0.0032 \\ -0.0002 & 0.0032 & 0.0448 \end{bmatrix} \quad \cdots \quad P_{25} = \begin{bmatrix} 0.0043 & 0.0020 & 0.0026 \\ 0.0020 & 0.0042 & 0.0059 \\ 0.0026 & 0.0059 & 0.0225 \end{bmatrix}$$

Therefore, the global asymptotic stability is assured. In order to apply conicity and harmonic balance equations, the affine system is split into a linear part and a non-linear part. The cone centre

$$\begin{bmatrix} 0.9946 & 0.0003 \\ 0.0003 & 1.0025 \end{bmatrix} \times 10^4$$

provides a conicity index of: 0.9965, and therefore, the system is stable (see Section 2).

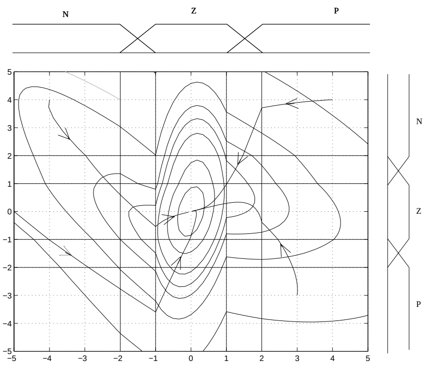

Figure 4. Globally stable system and Lyapunov level curves.

To apply the harmonic balance method, the dual describing function has been used, to deal with the non-symmetrical characteristic of the nonlinearities in the fuzzy control system. A solution exists if the two indexes, $I_1 = \det(G(0)N_0(a_0, a_1) + I)$ and $I_2 = \det(G(j\omega)N_1(a_0, a_1) + I)$, are zero. In this way, the results shown in Table 1 are obtained. Such results are congruent with the previous ones.

Example 2: TS Model + TS Controller

This example considers a TS model with a feedback TS controller without saturation in the control action. In this case the system is globally stable as can be verified by simulation (see Fig. 5). For this example a piecewise quadratic Lyapunov function was not found, and therefore no conclusions about stability could be done.

However, applying the conicity criteria or harmonic balance equations, the stability of the system can be stated. A conicity index of 0.9962 is obtained for the cone centre

$$\begin{bmatrix} 1.0029 & 0.0002 \\ 0.0002 & 0.9948 \end{bmatrix} \times 10^4 .$$

In the case of harmonic balance equations, stability can be also assured (see Table 1).

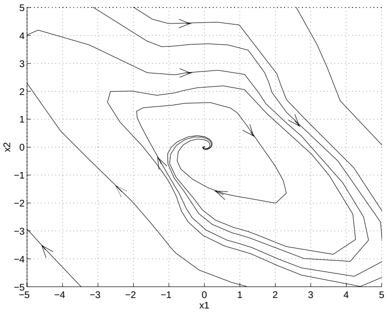

Figure 5. Globally stable system without saturation.

Example 3: Control action saturation

If a saturation in the control action exists (i.e., $-7.5 < u < 7.5$), an unstable limit cycles appears as shown in Fig. 6.

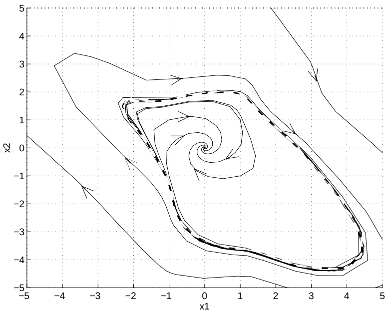

Figure 6. Unstable limit cycle due to saturation.

Using piecewise quadratic Lyapunov function or conicity criteria no stability results were obtained. However, using the harmonic balance equations a solution exists, and a limit cycle, with frequency $\omega = 3.7310$, is detected (see Table 1). Therefore the system is not globally stable. This example shows a case where frequency methods are substantially more powerful than the other methods.

Example 4: Saddle points

The last example corresponds to a system with two saddle points (see Fig. 7). Again, Lyapunov and conicity criteria can not provide stability conclusions. Nevertheless, a solution to the harmonic balance equation is obtained (see Table 1), where ω close to zero ($\omega = 0.0001$) is usually referred in the literature as a new equilibrium point.

Example	I_1	I_2	a_0^T	a_1^T	ω	Conclusion
1	3.2307	0.0277	[0.0127 0.4298]	[1.9870 0.4747]	0.8897	**Stable**
2	0.007	0.003	[0.4686 0.5019]	[0.9262 0.9846]	3.0701	**Stable**
3	4.3149×10^{-8}	2.6320×10^{-8}	[0.0117 0.0108]	[1.6031 1.0063]	3.7310	**Unstable Limit Cycle**
4	3.1465×10^{-12}	1.6839×10^{-7}	[0.5577 0.8535]	[0.9569 1.5991]	0.0001	**Multiple Equilib. Points**

Table 1: Results of Frequency Response analysis

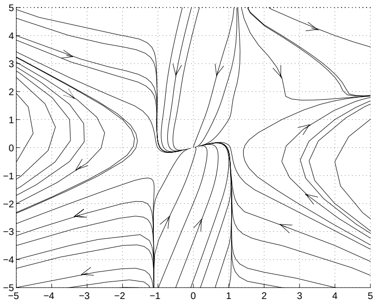

Figure 7. Multiple equilibrium points.

4. Conclusions

This paper has studied the stability of multivariable fuzzy control systems. Lyapunov approach is usually very conservative. Thus, it is not possible to find quadratic Lyapunov functions of many stable fuzzy control systems. Piecewise quadratic functions offer many solutions to stability problems that cannot be solved with conventional techniques.

Input-Output methods can also be straightforward applied to MIMO systems. The direct application of the small gain theorem leads to conservative stability results. The conicity criterion decreases conservatism, but this is still significant. These methods require training to select the involved parameters. Thus, several techniques have been implemented in FAMIMO to overcome this practical difficulty. Frequency response methods do not suffer of the same conservatism inherent to other stability analysis techniques. The application of classical frequency response techniques in multivariable systems is complex. However, the tools developed in FAMIMO can be applied without significant expertise.

In the paper some different approaches have been compared through particular examples. For local analysis around the operating point all the methodologies display similar results. However, for global problems (multiple equilibrium or limit cycles) the advantages of frequency methods have been pointed out. Thus, the comparative analysis of all the existing techniques to analyse the stability of multivariable fuzzy control systems, carried out in the FAMIMO project, reveals that there is no a single method better than the others. In fact several methods are complementary.

All the above techniques for stability analysis are being integrated in a Matlab Fuzzy Algorithm Stability Tool called FAST.

Acknowledgments

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