On the Robust Stability of 2-D Schur Polynomials

N. E. MASTORAKIS
Department of Computer Science,
Military Institutions of University Education,
Hellenic Naval Academy,
Terma Hatzikyriakou 18539,
Piraeus, GREECE,

Abstract : In this note, the problem of the robust stability for a two-dimensional (two-variable) Schur polynomial which is the characteristic polynomial of a discrete-time linear time-invariant system is investigated. A new approach based on the Rouché's theorem is adopted. The extension for multidimensional (multivariable) polynomials robust stability is also provided. Interesting sufficient conditions for such robust stability are derived. A two-dimensional example is included to support the theoretical result.

Key Words: Multidimensional Polynomial Theory, Robustness, Kharitonov's theorem, Stability, Schur polynomials, inverse Kharitonov's problem, Rouché's theorem.

1. Introduction and problem formulation

It is well known that a discrete variables' 2-D (two-dimensional) system must be Schur stable in order to ensure that any bounded input will produce a bounded output. Otherwise, the system is not useful for any practical application. Consider a shift-invariant, causal, single-input single-output, discrete variables' 2-D system described by the transfer function:

$$G(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)} \qquad (1)$$

$A(z_1, z_2)$ and $B(z_1, z_2)$ are coprime polynomials with real coefficients in the independent complex variables z_1 and z_2. It is assumed that there are no nonessential singularities of the second kind on the closed unit bidisk, i.e. there are no points (z_1,z_2) with $|z_1| \le 1$ and $|z_2| \le 1$ such that $A(z_1, z_2) = B(z_1, z_2)$=0. It is also well known – Huang's Theorem - that the system (1) is Schur stable if and only if

$$B(0, z_2) \ne 0, \qquad \text{for} \qquad |z_2| \le 1 \qquad (2.1)$$

$$B(z_1, z_2) \ne 0, \qquad \text{for} \qquad |z_1| \le 1, \qquad |z_2| = 1 \qquad (2.2)$$

Equation (2.1) is relatively easy to check using any 1-D stability test. Equation (2.2) is more difficult since it includes two variables. The polynomial $B(z_1, z_2)$ is said to be Schur stable if and only if (2.1) and (2.2) are fulfilled.

The problem of the robust stability of one-variable polynomials has attracted much attention in recent literature in which the most notable result is Kharitonov's theorem (Ref.1) and its various genaralizations. According to Kharitonov's theorem, a whole class of polynomials is Schur if and only if *four* special, well-defined polynomials are Schur polynomials. A great number of recent papers present various proofs, applications, generalizations and interesting results of this significant theorem (Refs. 1-5). In this note, the following two-dimensional robust stability problem is stated and discussed:

Suppose that the polynomial

$$f(z_1,z_2)=\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} a_{i_1,i_2} z_1^i z_2^j$$

$$a_{i_1,i_2}\in \mathbf{R},\ z_1,z_2\in \boldsymbol{C} \qquad (3)$$

is Schur stable that is (2.1) and (2.2) are fulfilled i.e. $f(0,z_2)\neq 0,$ for $|z_2|\leq 1$ and $f(z_1,z_2)\neq 0,$ for $|z_1|\leq 1,$ $|z_2|=1$. *Find* the maximum *R* (*R*>0), such that all the polynomials $g(z_1,z_2)=\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} b_{i,j} z_1^i z_2^j$ with $b_{i,j}\in\left[a_{i,j}-R, a_{i,j}+R\right],$ $i=0,1,\ldots,N_1$ $j=0,1,\ldots,N_2$ to be Schur stable.

2. Robust Stability for Two-Dimensional Polynomials

The approach which is attempted here is based on the following theorem, due to E. Rouché (Ref.6).

Rouché's theorem : If the two one-variable complex functions $f(z)$ and $g(z)$ are analytic in a region *G* of the complex plane and if

$$|g(z)-f(z)|<|f(z)| \qquad (4)$$

holds at every point *z* of the boundary ∂G of *G*, then the two functions $f(z)$ and $g(z)$ have the same numbers of zeros in *G*.

Now, one can easily prove the following Theorem.

Theorem 1: Consider the two two-variable complex polynomials $f(z_1,z_2)$ and $g(z_1,z_2)$ in the unit bi-disk of the complex bi-plane. If

$$|g(0,z_2)-f(0,z_2)|<|f(0,z_2)| \quad |z_2|=1 \qquad (5.1)$$

$$|g(z_1,z_2)-f(z_1,z_2)|<|f(z_1,z_2)| \quad |z_1|=|z_2|=1 \qquad (5.2)$$

and $f(z_1,z_2)$ is Schur stable, then $g(z_1,z_2)$ is also Schur stable.

Proof: It is based on Huang's and Rouche's Theorem and is omitted for the sake of brevity

Using Theorem 1, one can easily find the following sufficient condition for the inequality (5.1)

$$\sum_{j=0}^{N_2}|a_{0,j}-b_{0,j}||z_2^j|<|f(0,z_2)| \qquad (6.1)$$

or , since $|z_2|=1$, the following sufficient condition can be formulated

$$(N_2+1)\cdot R<|f(0,z_2)| \quad \text{with} \quad |z_2|=1$$

from which one finds the condition

$$R<\frac{\min|f(0,z_2)|\big|_{|z_2|=1}}{N_2+1} \qquad (7.1)$$

Using Theorem 1, one can easily find the following sufficient condition for the inequality (5.2)

$$\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}|a_{i,j}-b_{i,j}||z_1^i z_2^j|<|f(z_1,z_2)| \qquad (6.2)$$

or , since $|z_1|=|z_2|=1$, the following sufficient condition can also be formulated

$$(N_1+1)(N_2+1)R<|f(z_1,z_2)| \quad \text{with} \quad |z_1|=|z_2|=1$$

from which one finds the condition

$$R < \frac{\min \left| f(z_1, z_2) \right|_{|z_1|=|z_2|=1}}{(N_1+1)(N_2+1)} \tag{7.2}$$

So, a sufficient condition for (5.1) and (5.2) to hold simultaneously is

$$R < \min \left\{ \frac{\min \left| f(0, z_2) \right|_{|z_2|=1}}{N_2+1}, \frac{\min \left| f(z_1, z_2) \right|_{|z_1|=|z_2|=1}}{(N_1+1)(N_2+1)} \right\} \tag{8}$$

Therefore the calculation of an R that satisfies (8) includes one 1-dimenional and one 2-dimensional minimization problem. Both of them can be solved numerically. This minimization is actually achieved by minimizing the functions $\left| f(0, z_2) \right|^2_{|z_2|=1}$ and $\left| f(z_1, z_2) \right|^2_{|z_1|=|z_2|=1}$

These two functions are differentiable, since after some algebraic manipulation, it is verified that

$$\left| f(0, z_2) \right|^2_{|z_2|=1} = \sum_{j=0}^{N_2} a_{0,j}^2 + 2\sum_{l=1}^{N_2} \sum_{j=0}^{N_2-l} a_{0,j} a_{0,j+l} \cos(l q_2) \tag{9.1}$$

and

$$\left| f(z_1, z_2) \right|^2_{|z_1|=|z_2|=1} = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} a_{i,j}^2 + \\ + 2 \sum_{\substack{k=0 \\ (k,l)\neq(0,0)}}^{N_1} \sum_{l=0}^{N_2} \sum_{i=0}^{N_1-k} \sum_{j=0}^{N_2-l} a_{i,j} a_{i+k,j+l} \cos(k q_1 + l q_2) \tag{9.2}$$

One can also write them in the following form:

$$f(z) f(z^{-1}) = \left[\sum_{i=o}^{n} a_i^2 \quad \middle| 2\sum_{i=0}^{n-1} a_i a_{i+1} \quad \middle| \cdots \middle| 2\sum_{i=0}^{n-k} a_i a_{i+k} \middle| \cdots \middle| 2\sum_{i=0}^{n-n} a_i a_{i+N_2} \right].$$

$$\begin{bmatrix} 1 \\ \cos q \\ \vdots \\ \cos kq \\ \vdots \\ \cos nq \end{bmatrix} \tag{10.1}$$

and

$$\left| f(z_1, z_2) \right|^2_{|z_1|=|z_2|=1} = \mathrm{Re}\{ \begin{bmatrix} 1 & \cdots & e^{jkq_1} & \cdots & e^{jN_1 q_1} \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} a_{i,j}^2 & \cdots & 2\sum_{i=0}^{N_1-0} \sum_{j=0}^{N_2-N_2} a_{i,j} a_{i,j+l} \\ \vdots & 2\sum_{i=0}^{N_1-k} \sum_{j=0}^{N_2-l} a_{i,j} a_{i+k,j+l} & \vdots \\ 2\sum_{i=0}^{N_1-N_1} \sum_{j=0}^{N_2-0} a_{i,j} a_{i+k,j} & \cdots & 2\sum_{i=0}^{N_1-N_1} \sum_{j=0}^{N_2-N_2} a_{i,j} a_{i+N_1,j+N_2} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ e^{jlq_2} \\ \vdots \\ e^{jN_2 q_2} \end{bmatrix} \} \tag{10.2}$$

Therefore the functions $\left| f(0, z_2) \right|^2_{|z_2|=1}$ and $\left| f(z_1, z_2) \right|^2_{|z_1|=|z_2|=1}$ are differentiable functions in q_2 and q_1, q_2 and their minimum for $0 \le q_2 \le 2p$ and $0 \le q_1 \le 2p$, $0 \le q_2 \le 2p$ can be found using any numerical technique (Ref.7).

Furthermore, generalizing the above formulated problem one can give various weights to the interval of the coefficients $b_{i,j}$ i.e. $b_i \in [a_i - l_i R, a_i + l_i R]$ $l_i > 0$ $i = 0,1,\ldots,n$ $j = 0,1,\ldots,N_2$ and following the same steps one finds the following sufficient relation for robust stability

$$R < \min\left\{ \frac{\min\left| f(0,z_2)\right|\Big|_{|z_2|=1}}{\sum_{j=0}^{N_2} l_{0,j}}, \frac{\min\left| f(z_1,z_2)\right|\Big|_{|z_1|=|z_2|=1}}{\sum_{i=0}^{N}\sum_{j=0}^{N_2} l_{i,j}} \right\} \quad (11)$$

3. Robust stability for multivariable polynomials

An m-D (m-dimensional) system $G(z_1,\ldots,z_m)=\dfrac{A(z_1,\ldots,z_m)}{B(z_1,\ldots,z_m)}$ is Schur stable if and only if

$$B(0,\ldots,0,z_m)\neq 0, \quad \text{for} \quad |z_m|\leq 1 \quad (12.1)$$

$$B(0,\ldots,0,z_{m-1},z_m)\neq 0, \quad \text{for} \quad |z_{m-1}|\leq 1, \quad |z_m|=1 \quad (12.2)$$

$$\vdots$$

$$B(z_1,\ldots,z_{m-1},z_m)\neq 0, \quad \text{for} \quad |z_1|\leq 1,\ |z_2|=1\ ,\ldots,\ |z_m|=1 \quad (12.m)$$

when $A(z_1,\ldots z_m)$ and $B(z_1,\ldots z_m)$ are coprime polynomials with real coefficients in the independent complex variables z_1, z_2, …, z_m. It is also assumed that there are no nonessential singularities of the second kind on the closed unit m-disk, i.e. there are no points $(z_1,\ldots z_m)$ with $|z_1|\leq 1$, …, $|z_m|\leq 1$ such that $A(z_1,\ldots z_m)=B(z_1,\ldots z_m)$=0. ("m-dimensional Huang's Theorem")

The polynomial $B(z_1,\ldots z_m)$ is said to be Schur Stable if and only if (12.1) and …and (12.m) are fulfilled. In this paragraph, the extension of the previously presented 2-D inverse robust stability problem to m-D polynomials is examined. The problem is stated as follows:

Suppose that the polynomial $f(z_1,\ldots,z_m)=\sum_{i_1=0}^{N_1}\cdots\sum_{i_m=0}^{N_m} a_{i_1,\ldots,i_m} z_1^{i_1}\ldots z_m^{i_m}$ with $a_{i_1,\ldots,i_m}\in \mathbf{R}$, $z_1^{i_1}\ldots z_m^{i_m}\in \boldsymbol{C}$ is Schur stable that is (12.1) and … and (12.m) are fulfilled for $f(z_1,\ldots,z_m)$. *Find* the maximum R (R>0) such that all the polynomials $g(z_1,\ldots,z_m)=\sum_{i_1=0}^{N_1}\cdots\sum_{i_m=0}^{N_m} b_{i_1,\ldots,i_m} z_1^{i_1}\ldots z_m^{i_m}$ with $b_{i_1,\ldots,i_m}\in \left[a_{i_1,\ldots,i_m}-R, a_{i_1,\ldots,i_m\,i}+R\right]$, $i_1=0,1,\ldots,n \quad i_m=0,1,\ldots,N_m$ to be Schur stable.

Based on the Rouche's theorem as well as on the m-dimensional Huang's theorem, we can derive the following:

Theorem 2: Consider the two m-variable complex polynomials $f(z_1,\ldots,z_m)$ and $g(z_1,\ldots,z_m)$ in the unit m-disk of the complex bi-plane. If

$$\left|g(0,\ldots,0,z_m)-f(0,\ldots,0,z_m)\right|<\left|f(0,\ldots,0,z_m)\right| \quad \text{for} \quad |z_m|=1 \quad (13.1)$$

and

$$\left|g(0,\ldots,0,z_{m-1},z_m)-f(0,\ldots,0,z_{m-1},z_m)\right|<\left|f(0,\ldots,0,z_m)\right| \quad \text{for} \quad |z_{m-1}|=1,\ |z_m|=1 \quad (13.2)$$

and

$$\vdots$$

and

$$\left|g(z_1,\ldots,z_{m-1},z_m)-f(z_1,\ldots,z_{m-1},z_m)\right|<\left|f(z_1,\ldots,z_{m-1},z_m)\right| \quad \text{for} \quad |z_1|=1,\ldots,|z_m|=1 \quad (13.m)$$

and $f(z_1,\ldots,z_m)$ is Schur stable, then $g(z_1,\ldots,z_m)$ is also Schur stable.

Following, now, the same steps of the afore mentioned procedure and after some algebraic manipulation, the following relation can be derived:

$$R < \min\left\{ \frac{\min\left| f(0,...,0,z_m)\right|_{|z_m|=1}}{N_m+1}, \frac{\min\left| f(0,...,0,z_{m-1},z_m)\right|_{|z_{m-1}|=|z_m|=1}}{(N_{m-1}+1)(N_m+1)},..., \frac{\min\left| f(z_1,...,z_m)\right|_{|z_1|=...=|z_m|=1}}{(N_1+1)\cdots(N_m+1)} \right\} \tag{14}$$

The calculation of an R that satisfies (14) includes one 1-dimensional, one 2-dimensional, .. and one m-dimensional minimization problem. This minimization is actually achieved by minimizing the following differentiable functions

$$\left| f(0,...,0,z_m)\right|^2_{|z_m|=1}, \left| f(0,...,0,z_{m-1},z_m)\right|^2_{|z_{m-1}|=|z_m|=1} \quad ,..., \quad \left| f(z_1,...,z_m)\right|^2_{|z_1|=...=|z_m|=1}$$

These functions are differentiable, since after some algebraic manipulation, it is verified that, for every $n = 1,...m$, we have that:

$$\left| f(0,...,0,z_n, z_{n+1},...,z_m)\right|^2_{|z_1|=|z_2|=1} = \sum_{i_n=0}^{N_n} \sum_{i_{n+1}}^{N_{n+1}} \cdots \sum_{i_m=0}^{N_m} a^2_{0,...,0,i_n,i_{n+1},...,i_m} +$$

$$\sum_{k_n=0}^{N_n} \sum_{\substack{k_{n+1}=0 \\ (k_n,...,k_m)\neq(0,...,0)}}^{N_{n+1}} \cdots \sum_{k_m=0}^{N_m} \sum_{i_n=0}^{N_n-k_n} \sum_{i_{n+1}=0}^{N_{n+1}-k_{n+1}} \cdots \sum_{i_m=0}^{N_m-k_m} a_{0,..0,i_n,i_{n+1},...,i_m} a_{0,...,0,i_n+k_n,i_{n+1}+k_{n+1},...,i_m+k_m} \cos(k_n q_n + k_{n+1} q_{n+1} + k_m q_m)$$

Also, generalizing and giving various weights to the interval of the coefficients $b_{i,j}$ i.e. $b_{i_1,...,i_m} \in \left[a_{i_1,...,i_m} - l_{i_1,...,i_m} R, a_{i_1,...,i_m} + l_{i_1,...,i_m} R \right] \qquad l_{i_1,...,i_m} > 0$ $\quad i = 0,1,..., N_1 \qquad j = 0,1,..., N_2$ and following the same steps one finds the relation

$$R < \min\left\{ \frac{\min\left| f(0,...,0,z_m) \right|\Big|_{|z_m|=1}}{\sum_{i_m}^{N_m} l_{0,...,0,i_m}}, ..., \frac{\min\left| f(0,...,0,z_n, z_{n+1},...z_m) \right|\Big|_{|z_n|=...=|z_m|=1}}{\sum_{i_n}^{N_n} \cdots \sum_{i_m}^{N_m} l_{0,...,0,i_n,...,i_m}}, ..., \frac{\min\left| f(z_1,...z_m) \right|\Big|_{|z_1|=...=|z_m|=1}}{\sum_{i_1}^{N_1} \cdots \sum_{i_m}^{N_m} l_{i_n,...,i_m}} \right\}$$

(11)

4. Numerical example

Consider the polynomial

$f(z_1, z_2) = 2 + z_1 + z_2 + z_1 z_2$. This is a (Schur) stable polynomial since it fulfils the equations (2.1) and (2.2). Furthermore, we can find the maximum R such that all the polynomials

$g(z_1, z_2) = a + bz_1 + cz_2 + dz_1 z_2$ with $a \in [2 - R, 2 + R]$, $b \in [1 - R, 1 + R]$, $c \in [1 - R, 1 + R]$, $d \in [1 - R, 1 + R]$ to be stable. Applying (8) - or (14) in the special case with m=2 - one obtains

$$R < \min\left\{ \frac{\min\left| f(0, z_2) \right|\Big|_{|z_2|=1}}{N_2 + 1}, \frac{\min\left| f(z_1, z_2) \right|\Big|_{|z_1|=|z_2|=1}}{(N_1 + 1)(N_2 + 1)} \right\}$$

Since

$$\frac{\min\left| f(0, z_2) \right|\Big|_{|z_2|=1}}{N_2 + 1} = \frac{\min\left| 2 + z_2 \right|\Big|_{|z_2|=1}}{2} = \frac{1}{2}$$

and

$$\frac{\min\left| f(z_1, z_2) \right|\Big|_{|z_1|=|z_2|=1}}{(N_1 + 1)(N_2 + 1)} = \frac{\min\left| 2 + z_1 + z_2 + z_1 z_2 \right|\Big|_{|z_1|=|z_2|=1}}{4} = \frac{\sqrt{2}/2}{4} = 0.1767$$

one easily obtains $R < 0.1767$. For example R = 0.1761

4. Conclusion

The problem of the robust stability for a two-dimensional and multidimensional (Schur) stable polynomial is investigated. An estimation for the range of the Schur stability has been given and has been illustrated by a numerical example. The applications of the present method in 2-D and m-D digital filters design are important for digital image processing and image enhancement.

References:

[1]. KHARITONOV V.L., *Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations*, Differential Equations, Vol. 14, pp.1483-1485, 1979.

[2]. CHAPELLAT H. and BHATTACHARYYA S.P., *An Alternative Proof of Kharitonov's Theorem*, IEEE Transactions on Automatic Control, Vol.34, No.4, pp.448-450, 1989.

[3]. MINNICHELLI R.J., ANAGNOST J.J. and DESOER C.A., *An Elementary Proof of Kharitonov's Stability Theorem*, IEEE Transactions on Automatic Control, Vol.34, No.9, pp.995-998, 1989.

[4]. TEMPO R., *A Dual Result to Kharitonov's Theorem*, IEEE Transactions on Automatic Control, Vol.35, No.2, pp.195-198, 1990.

[5]. XU S.J., DAROUACH M. and SCHAEFERS J., *Expansion of det(A+B) and Robustness Analysis of Uncertain State Space Systems*, IEEE Transactions on Automatic Control, Vol.38, No.11, pp.1671-1675, Nov.1993.

[6]. CARATHEODORY C., *Theory of Functions of a Complex Variable*, Chelsea Publishing Company, New York 1964.

[7]. LUENBERGER D.G., *Introduction to Linear and Nonlinear Programming*, Addison-Wesley Publishing Company, Reading, Massachusetts, 1973