Abstract: The locally-topological method for nonlinear dynamics analysis of signals is presented. The theoretic ground of a locally-topological method for determining a minimum attractor embedding dimension on the basis of state-space method is proposed.

Key Words: complex systems, attractor, minimal embedding dimension, state-space method

1 Introduction
In recent years the nonlinear dynamic systems (NDS) with selforganization named complex systems [1]-[4] are investigated with great activity. I. Prigogine and H. Haken have been established [1],[2] that functioning complex NDS is closely connected with the presence of chaos at their behaviour. We can describe the NDS behaviour based on the construction of system attractor in \( m \)-dimension Euclidean phase space \( \mathbb{R}^m \). It is necessary to select the phase space with the minimal dimension \( m_o \) as the value \( m_o \) is upper limit of the freedom degrees for system and hence \( m_o \) gives the minimal number of differential equations for the NDS modeling. The determing \( m_i \) on the basis of various correlative topological methods requires large computer expenditures and significant volume of experimental data [4]. The method proposed in [5]-[7] permits to reduce the computational complexity and the required experimental data and thus to decrease the lacks of correlative topological methods.

2 Theoretical Estimation of Minimal Attractor Dimension for Discrete Linear Dynamic System
Let us use the state-space method [8] for the dynamic system description. We will consider the finite-dimensional stationary discrete linear dynamic system (LDS) with one input \( x_n \) and output \( y_n \) in order that to simplify the calculations. In terms of the state-space this system is described by means of the relations [8],[9]:

\[
\hat{u}_{n+1}^{(m)} = K_m \hat{u}_n^{(m)} + b^{(m)} x_n; \quad y_n = c^{(m)T} \hat{u}_n^{(m)},
\]

where \( \hat{u}_n^{(m)} \in U \) is the state-space for the LDS of the dimension \( m \), i.e. \( U = \mathbb{R}^m \);

\[
K_m = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-a_m & -a_{m-1} & -a_{m-2} & \ldots & -a_1
\end{bmatrix}
\]

\[
E^{(m)} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
b_m
\end{bmatrix}; \quad c^{(m)T} = [1 \ 0 \ \ldots \ 0]_p
\]

\( T \) is transposition symbol, \( b_m = \frac{1}{a_0} \), \( a_0 \) is the coefficient at the higher term of the equation describing discrete stationary LDS.

It was shown in [9] that state-space \( U \) is \( K_m \)-stationary (that is invariant relatively to matrix operator \( K_m \) [9]-[11]) for any stationary LDS (both continuous and discrete). That is why the state-space \( U \) is cyclic vector space [9]. For \( x_n = 0 \) the equations (1) describe evolution of stationary discrete LDS in \( m \)-dimension state-space:

\[
\hat{u}_{n+1}^{(m)} = K_m \hat{u}_n^{(m)}; \quad y_n = c^{(m)T} \hat{u}_n^{(m)}. \tag{3}
\]

For calculating a distance between the neighbouring vectors in state-space \( U = \mathbb{R}^m \) let us use Euclidean metric \( l_2 \):
\[ d(\hat{u}_n^{(m)}, \hat{u}_{n+1}^{(m)}) = \| \Delta \hat{u}_n^{(m)} \| = \| \hat{u}_n^{(m)} - \hat{u}_{n+1}^{(m)} \| = \| (K_m - E_m) \hat{u}_n^{(m)} \| \leq \| K_m - E_m \| \| \hat{u}_n^{(m)} \| , \]

(4)

where \( E_m \) is the identity matrix. By analogy with (4) let us calculate:

\[ d(u_n^{(m)}, u_{n+1}^{(m)}) = \| u_n^{(m)} - u_{n+1}^{(m)} \| \leq \| K_m - E_m \| \| u_n^{(m)} \| \]

(5)

Starting from (1)-(5) let us estimate the relative distance \([5]-[7]\):

\[ p_n^{(m)} = \frac{d(\hat{u}_n^{(m)}, \hat{u}_{n+1}^{(m)})}{d(u_n^{(m)}, u_{n+1}^{(m)})} = \| \hat{u}_n^{(m)} - \hat{u}_{n+1}^{(m)} \| \leq \| K_m - E_m \| \| u_n^{(m)} \| \leq \| K_m \| \]

(6)

Calculating the Euclidean \( l_2 \)-norm \([12]\) for Frobenius matrix \( K_n \) we obtain:

\[ p_n^{(m)} \leq \left( m - 1 + \sum_{i=1}^{m} |\alpha_i|^2 \right)^{\frac{1}{2}}, \]

(7)

where \( \alpha_i = \frac{a_i}{a_b} \) are parameters of LDS in \( m \)-dimension state-space. As follows from (7), in state-space \( U = R^{m+1} \) the relative distance is estimated by means of:

\[ p_n^{(m+1)} \leq \left( m + \sum_{i=2}^{m+1} |\beta_i|^2 \right)^{\frac{1}{2}}, \]

(8)

where \( \beta_i \) are the parameters of LDS in \( (m+1) \)-dimension state-space. We can estimate the dynamics of a variation for a topological structure of attractor for \( R^m \rightarrow R^{m+1} \) by means of the relation:

\[ q_n^{(m,m+1)} = \frac{p_n^{(m+1)}}{p_n^{(m)}} \leq \left( m - 1 + \sum_{i=2}^{m+1} |\beta_i|^2 \right)^{\frac{1}{2}} \left( m - 1 + \sum_{i=1}^{m} |\alpha_i|^2 \right)^{\frac{1}{2}}. \]

(9)

One can see from (9) that if \( m = m_b \) is the dimension of state-space \( U \) for given LDS so under the transfer to \( (m+1) \)-dimension state-space the following relations take place:

\[ f : R^m \rightarrow R^{m+1} \Rightarrow \beta_{i+1} \rightarrow \alpha_i, i = 1, \ldots, m; \beta_1 \rightarrow 1. \]

(10)

With regard to (9) we can write:

\[ s = m - 1 + \sum_{i=1}^{m} |\alpha_i|^2. \]

(11)

Taking into account (10), (11) we can write the relation (9) by the following way:

\[ q_n^{(m,m+1)} \leq \left( s + \beta_1^2 + 1 \right)^{\frac{1}{2}} \left( 1 + \frac{2}{s} \right)^{\frac{1}{2}}. \]

(12)

For large \( s \) the value \( q_n^{(m,m+1)} \rightarrow 1 + 1/s \rightarrow 1 \). It should be noted that for not large \( s \), i.e. for low-dimensional LDS the value \( q_n^{(m,m+1)} \) is stabilized to some \( q_n^{(m,m+1)} \geq 1 \). However in the case of large values of parameters \( (\alpha_i \rightarrow \infty) \) even for low-dimensional LDS stabilization \( q_n^{(m,m+1)} \rightarrow 1 \) takes place too. For low-dimensional LDS \( (m \not= too \ large) \) and small values of LDS parameters \( (|\alpha_i| \not= small \ quantities) \) the stabilization \( q_n^{(m,m+1)} \) tends to the value \( q_n^{(m,m+1)} \not= 1 \). Thus it is appropriately to introduce value \( q_n^{(m,m+1)} \) averaged over point set of attractor - so called function of topological nonstability \( Z_c \) \([5]-[7]\). Taking into account (12), we can see that the function of topological nonstability \( Z_c \) should be constant \( Z_c \rightarrow 1 \).

### 3 Estimation of Minimal Attractor Dimension for Discrete LDS with an Input Action

Now let us consider finite-dimensional stationary discrete LDS with nonzero input action \( x_c \). Let us estimate the distance between the neighbouring vectors in state-space \( U = R^n \). Taking into account (1) let us write:

\[ d(u_n^{(m)}, u_{n+1}^{(m)}) = \| K_m - E_m \| u_n^{(m)} + b^{(m)} x_c \| \leq \| K_m - E_m \| u_n^{(m)} + \| b^{(m)} \| x_c, \]

(13)

where \( b_m = \frac{1}{a_b} \), \( a_b \) is the parameter of \( m \)-dimensional stationary discrete LDS, i.e. the coefficient at higher term for the difference or (in the case of continuous LDS) for differential equation. By analogy with (13) one can see that

\[ d(\hat{u}_n^{(m)}, \hat{u}_{n+1}^{(m)}) \leq \| K_m - E_m \| \| \hat{u}_n^{(m)} + b^{(m)} \| x_c \| \]

(14)

With regard to (6), (11), (13), (14) we can estimate the relative distance \([5]-[7]\) by means of the following relation:

\[ p_n^{(m)} \leq \frac{\| K_m - E_m \| \| u_n^{(m)} + b^{(m)} \| x_c \|^2 \leq \| K_m - E_m \| \| u_n^{(m)} \| x_c \| \}

(14)

\[ p_n^{(m)} \leq \frac{\| K_m - E_m \| \| \hat{u}_n^{(m)} + b^{(m)} \| x_c \|^2 }{\| K_m - E_m \| \| \hat{u}_n^{(m)} \| x_c \|} \]

(12)
\[ s^2 \leq \frac{\left| a_n^m + b_n^m x_n^2 + b_n^m x_n^2 \right|^2 / (m - 2 x_n + s)^2}{\left| a_n^m + b_n^m x_n^2 + b_n^m x_n^2 \right|^2 / (m - 2 x_n + s)^2}, \]  

(15)

where \( \alpha_i = \frac{a_i}{a_0} \) are the parameters of matrix LDS in state-space \( U = R^m \). If \( m = m_0 \) is the dimension of state-space \( U \) for LDS under consideration, its mapping into \( (m + 1) \)-dimensional space makes correct the relation (10) as well as the following to (11) and (15) the value \( q_n^{(m, m+1)} \) can be estimated by the next relation

\[ q_n^{(m, m+1)} \leq \frac{(s + 2)^2}{s^2} \left[ a_n^m + b_n^m x_n^2 + b_n^m x_n^2 \right]^2 / (m - 2 x_n + s)^2 \]  

(17)

As we can see from (17) the term \( \left| b_n^m \right|^2 \) can have large value for the external action as a jump function. Really, for large \( s \) the value \( q_n^{(m, m+1)} \) will tend to 1 as it follows from the arguments with respect to (12). But if the action \( x_n \) has high amplitude at the moment \( n \), the stabilization \( \hat{q}_n^{(m, m+1)} \) can be broken in view of the fact that

\[ q_n^{(m, m+1)} << 1 \]  

(18)

for the large \( s \) and sharp increase of the action \( x_n \) at the moment \( n \). In this case the averaged upon the point set of attractor value \( \bar{a}_n^{(m, m+1)} = Z(m) \) is destabilized. If we represent function \( Z(m) \) as graph, one can see the typical "drop of stabilization" at the moments corresponding to "splash of amplitude" of the external action \( x_n \).

4 Minimal Attractor Dimension
Estimation for NDS

For the analysis of discrete NDS described in state-space by the relations

\[ \tilde{u}_{n+1} = \tilde{f}_1(\tilde{u}_n, x_n), \quad \tilde{y}_n = \tilde{f}_2(\tilde{u}_n, x_n), \]  

(19)

(where \( \tilde{f}_1(\cdot) \), \( \tilde{f}_2(\cdot) \) are some nonlinear functions), the proposed above method for determining a minimal attractor embedding dimension can be used too. In view of the fact that the relation (1) is generalized by (19), the theoretic verification of this method for NDS is to linearize the function \( \tilde{f}_1(\cdot) \) near zero state \( \Omega^* \) and to apply all arguments with respect to discrete LDS. For NDS’s case we have to use nonlinear decomposition of the vector function \( \tilde{f}_1(\cdot) \) in state-space (see [13]) and to carry out analogous derivation according (1)-(18).

As stated in [13] a change of vector-function in state-space is decomposed in vector-matrix series of the kind:

\[ \Delta \tilde{f}(\tilde{v}) = \tilde{f}(\tilde{u}^* + \tilde{v}) - \tilde{f}(\tilde{u}^*) = L_m^{(1)} \tilde{v} + \frac{1}{2!} L_m^{(2)} (\tilde{v} \otimes \tilde{v}) + \frac{1}{3!} L_m^{(3)} (\tilde{v} \otimes \tilde{v} \otimes \tilde{v}) + \ldots \]  

(20)

where

\[ L_m^{(1)} = \frac{\partial}{\partial \tilde{v}} \tilde{f}(\tilde{u}^*) \]  

\[ L_m^{(2)} = \frac{\partial}{\partial \tilde{v}} \left( \tilde{f}(\tilde{u}^*) \right) \]  

\[ L_m^{(3)} = \frac{\partial}{\partial \tilde{v}} \left( \tilde{f}(\tilde{u}^*) \right) \]  

\[ \otimes \] denotes symbol of Kronecker matrix product.

According to (20) we study behavior of the decision \( u \) of the equation (19) near to a specific standard state \( \Omega^* \) being considered as the undisturbed decision according (19), permanently disturbed by external actions or internal fluctuations on value \( V \) [1]. In result instead of \( \Omega^* \) there is a new decision \( \Omega = \Omega^* + V, ]M_{\Omega^*} [V] \) <1. In view of this one rewrites the relation (20) as follows:

\[ \tilde{f}(\tilde{u}) = \tilde{f}(\tilde{u}^*) + L_m^{(1)} (\tilde{u} - \tilde{u}^*) + \frac{1}{2!} L_m^{(2)} (\tilde{u} - \tilde{u}^*) \otimes (\tilde{u} - \tilde{u}^*) + \frac{1}{3!} L_m^{(3)} (\tilde{u} - \tilde{u}^*) \otimes (\tilde{u} - \tilde{u}^*) \otimes (\tilde{u} - \tilde{u}^*) + \ldots \]  

(21)

Let us estimate the distance (4) and (5) between the neighbouring vectors in state-space \( U = R^m \) using linearization of function \( \tilde{f}_1(\tilde{u}_n, x_n) \) in (19) according to (21):

\[ d(\tilde{u}_n^{(m)}, \tilde{u}_n^{(m)}) = \left\| \tilde{u}_n^{(m)} - \tilde{u}_n^{(m)} \right\| = \left\| \tilde{f}_1(\tilde{u}_n^{(m)}, x_n) - \tilde{u}_n^{(m)} \right\| = \left\| \tilde{f}_1(\tilde{u}_n^{(m)}, x_n) \right\| \approx \left\| \tilde{f}(\tilde{u}_n^{(m)}, x_n) \right\| + L_m^{(1)} (\tilde{u}_n^{(m)} - \tilde{u}_n^{(m)}) + \ldots + \tilde{u}_n^{(m)} \right\| < \left\| L_m^{(1)} - E_m \right\| \tilde{u}_n^{(m)} + \tilde{f}(\tilde{u}_n^{(m)}, x_n) - L_m^{(1)} \tilde{u}_n^{(m)} \right\| \]  

(22)
\[
d(d(u_{n1}^{(m)}, u_{n2}^{(m)}) = \|u_{n2}^{(m)} - u_{n1}^{(m)}\| =
\]
\[
= \left\| f(1(u_{n1}^{(m)}, x_n) - u_{n1}^{(m)}) \right\| \leq \|f(1(u_{n1}^{(m)}, x_n) - L(1)_{m}u_{n1}^{(m)})\| +
\]
\[
+ \|f(u_{n1}^{(m)}, x_n) - L(1)_{m}u_{n1}^{(m)}\|.
\]

Taking into account (21) and (23) then we can calculate the relative distance (6) by means of :
\[
p_n^{(m)} = \left\| L(1)_{m} - E_{m} \right\| \left\| L(1)_{m}^{-1} + \|f(1(u_{n1}^{(m)}, x_n) - L(1)_{m}u_{n1}^{(m)})\| \leq
\]
\[
\leq \left\| f(u_{n1}^{(m)}, x_n) + L(1)_{m}(u_{n1}^{(m)} - u_{n1}^{(m)}) + \ldots + \left\| f(u_{n1}^{(m)}, x_n) - L(1)_{m}u_{n1}^{(m)}\right\| \|L(1)_{m} - E_{m}\|ight.
\]
\[

Thus, the estimation of the \( q_n^{(m,m+1)} \) when \( R^m \rightarrow R^{m+1} \) requires more generalized condition than (10) which leads to the relation (12) only in some cases.

References: