Forward Displacement Analysis of Two Classes of Stewart Platform Using One Unified Mathematical Model

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Abstract  
In this paper, we studied the forward analysis of two classes of Stewart platform using a unified mathematical model. The kinematic equations are expressed as a system of nine quadratic equations in nine unknowns. Solution of this system of equations can be reduced to solving a system of six second order polynomials in six unknowns. Empirical experiment shows that for generic parameters the problem has 28 solutions. Since the system has certain symmetry, we need only to find 14 of the 28 solutions. Further, for any given platform, an efficient nonlinear homotopy can be used to identify the 14 solutions by following 14 continuation curves and the average CPU time on a Sun Sparc Station ECL in double precision complex arithmetic is less than 3 seconds.

Key Words: Continuation Method, Crossed Type Stewart Platform, Forward Displacement Analysis, Polynomial equation, Robotics

1 Introduction  
During the past few years, parallel manipulator systems have become one of the research attentions in robotics since such a system can offer high stiffness, high load capability, quick dynamic response and good positioning accuracy. Among the many parallel manipulators, the Stewart platform [1] and its many variations are probably the most common ones used in both analysis and applications. The original Stewart platform was introduced as an aircraft simulator by Stewart in 1965. The most general form of the Stewart Platform, as shown in Fig. 1, consists of two platforms which are connected to each other by six legs. Each leg has a spherical joint at one end and a universal joint (or spherical joint) at the other. These joints are passive. There is an actively controlled prismatic joint between the two ends of each leg. One of the platforms, called the “top platform”, has six degrees of freedom relative to the other platform which is usually referred to as the “base platform”.

Among the many aspects of parallel robotic systems, direct kinematics has been studied extensively. As opposed to serial manipulators, direct kinematics of parallel manipulators is more complicated than the inverse kinematics. Direct kinematics of Stewart Platform is important when it is controlled by a cartesian controller, or it is used as a force/torque sensor or the master manipulator in a teleoperation system. This problem has been studied by many researchers. Namu et al. [2] presented a solution for the forward position kinematics of a special form of the 6 DOF Stewart Platform and obtained an 8-th order polynomial in a single variable. Lin et al. [3] considered the forward position problem of the 4-4 Stewart Platform, in which two pairs of joints are coincident in both the base and the moving platform, and obtained an 8-th order and a 12-th order polynomials for different cases. Later, Zhang and Song [4] studied the forward position analysis of general Stewart Platform, and derived a closed-form solution. It was shown that the resultant equation is a 20-th order polynomial.

Recently, homotopy continuation method has been used for solving kinematics problems. Tsai and Morgan [5] used homotopy homotopy continuation method for solving the inverse kinematics problem of 6R serial manipulators. Later Wampler and Morgan [6] also used continuation method for inverse kinematics analysis of the 6R manipulator. For the first time, Raghavan [7] applied continuation method to the forward position analysis of the Stewart Platform and found that there are forty possible configurations. The formulation used by Raghavan resulted in a system of nine unknowns. Then, multi-homogeneous coordinates were used to reduce the Bézout number to 960. Hence 960 paths had to be tracked to obtain the 40 finite solutions of the general Stewart Platform problem.

In this paper, we consider two classes of 6-6 Stewart Platform in which the six joints in both the base and the top platforms located in two planes and arranged in semi-regular hexagons. The first case, shown in Fig.
4, is referred to as the Non-crossed Type and is one of the most popular parallel mechanisms in practice. The second case, shown in Fig. 2, differs from the former in geometric structure. The top and base platforms are connected by three pairs of crossed legs. Hence, it is referred to as the Crossed Type Stewart Platform. It will be shown that the direct kinematics of these two classes of Stewart Platform can be reduced to a system of six second degree polynomial equations in six unknowns and an efficient numerical continuation method can be applied for solving this problem. It was found that this system has exactly 28 distinct, finite solutions. Therefore, each of these two classes of Stewart platform has at most 28 configurations.

2 Modeling

2.1 Crossed Type Stewart Platform

The direct kinematic problem of Stewart platform can be stated as follows: Given the lengths of the six legs, find the position and orientation of the top platform relative to the base platform.

Attach a fixed Cartesian coordinate frame $O-XYZ$ to the base platform with the $Z$ axis pointing vertically upwards and a moving Cartesian coordinate $O_1-X_1Y_1Z_1$ to the top platform with $Z_1$ axis normal to the platform. Let the homogeneous transformation matrix representing the position and orientation of the top platform relative to the base platform be

$$
\begin{bmatrix}
x_1 & x_4 & x_7 & x_{10} \\
x_2 & x_5 & x_8 & x_{11} \\
x_3 & x_6 & x_9 & x_{12} \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

Let the six vertices of the top platform relative to $O_1-X_1Y_1Z_1$ be

$$P_i (P_{ix}, P_{iy}, 0, 1), \ i = 1, \ldots, 6,$$  

then the coordinates of $P_i$ relative to $O-XYZ$ are

$$
\begin{align*}
(P_{ix}x_1 + P_{iy}x_4 + x_{10}, & \ P_{ix}x_2 + P_{iy}x_5 + x_{11}, \\
 P_{ix}x_3 + P_{iy}x_6 + x_{12}, & \ i = 1, \ldots, 6.
\end{align*}
$$

Let the six vertices of the base platform relative to $O-XYZ$ be

$$B_i (B_{ix}, B_{iy}, 0, 1), \ i = 1, \ldots, 6.$$  

Using the distance formula in $\mathbb{R}^3$ and the properties of a homogeneous transformation matrix, we have the following equations

$$
\begin{align*}
(p_i x_1 + p_i y_4 + x_{10} - b_{ix})^2 + (p_i x_2 + p_i y_5 + x_{11} - b_{iy})^2 + (p_i x_3 + p_i y_6 + x_{12})^2 & = l_i^2, \\
x_1^2 + x_2^2 + x_3^2 & = 1, \\
x_4^2 + x_5^2 + x_6^2 & = 1, \\
x_1x_4 + x_2x_5 + x_3x_6 & = 0,
\end{align*}
$$

where $L_i$ is the length of the $i$-th leg which connects the joints $P_i$ and $B_i$.

$$
\begin{align*}
P_{ix} &= R\sin(\beta), & P_{iy} &= -R\cos(\beta), \\
P_{2x} &= R\cos(\pi/6 - \beta), & P_{2y} &= R\sin(\pi/6 - \beta), \\
P_{3x} &= R\sin(\pi/3 - \beta), & P_{3y} &= R\cos(\pi/3 - \beta), \\
P_{4x} &= -P_{3x}, & P_{4y} &= P_{3y}, \\
P_{5x} &= -P_{2x}, & P_{5y} &= P_{2y}, \\
P_{6x} &= -P_{1x}, & P_{6y} &= P_{1y},
\end{align*}
$$
$B_{1z} = R_0 \sin(\beta_0)$, $B_{1y} = -R_0 \cos(\beta_0)$,
$B_{2z} = R_0 \cos(\pi/6 - \beta_0)$, $B_{2y} = -R_0 \sin(\pi/6 - \beta_0)$,
$B_{3z} = R_0 \sin(\pi/3 - \beta_0)$, $B_{3y} = -R_0 \cos(\pi/3 - \beta_0)$,
$B_{4x} = -B_{3x}$, $B_{4y} = B_{3y}$,
$B_{5x} = B_{3x}$, $B_{5y} = B_{3y}$,
$B_{6x} = -B_{1x}$, $B_{6y} = B_{1y}$.

$R$ and $R_0$ represent the radii of top and base platforms respectively, $\beta(-\pi/3 < \beta < 0)$ and $\beta_0(0 < \beta_0 < \pi/3)$ are the half angles of $\angle P_0 P_3$ and $\angle P_0 P_3$. Using (6), (7), (8), $T_{ex}^2 + T_{ey}^2 = R^2$, and $T_{ex}^2 + T_{ey}^2 = R_0^2$, Eqs. (5) can be reduced to the following:

$$(L_1^2 + L_0^2)/2 = R^2 + R_0^2 + w - 2B_{1x}x_11 + 2P_{3y}w - 2(P_{1y}B_{1x}x_1 + P_{3y}B_{1y}x_5),$$

$$(L_2^2 - L_0^2)/4 = -B_{1x}x_{10} - P_{1y}B_{1y}x_2 - P_{1y}B_{1x}x_4 + P_{1x}u,$$ 

$$(L_2^2 + L_0^2)/2 = R^2 + R_0^2 + w - 2B_{2x}x_{11} + 2P_{2y}w - 2(P_{2y}B_{2x}x_1 + P_{3y}B_{2y}x_5),$$

$$(L_3^2 - L_0^2)/4 = -B_{2x}x_{10} - P_{2y}B_{2y}x_2 + P_{2y}B_{2x}x_4 - P_{2x}u,$$ 

$$(L_3^2 + L_0^2)/2 = R^2 + R_0^2 + w - 2B_{3x}x_{11} + 2P_{3y}w - 2(P_{3y}B_{3x}x_1 + P_{3y}B_{3y}x_5),$$

$$(L_3^2 - L_0^2)/4 = -B_{3x}x_{10} - P_{3y}B_{3y}x_2 + P_{3y}B_{3x}x_4 + P_{3x}u,$$

where

$$u = x_1x_{10} + x_2x_{11} + x_3x_{12},$$

$$v = x_1x_5 + x_2x_{11} + x_6x_{12},$$

$$w = x_{10}^2 + x_{11}^2 + x_{12}^2.$$ 

Considering system (9)-(14) as linear equations in $x_1, x_2, x_4, x_5, u, v$ and $w$, the coefficient matrix is singular if and only if $\beta = \beta_0$. For the Non-crossed Type Stewart Platforms, $\beta \neq \beta_0$. So $x_1, x_2, x_4, x_5, u, v$ and $w$ can be expressed in terms of $x_{10}, x_{11}, x_{12}, w$.

$$D_2^2[A_2^2x_{10}^2 + 2A_1x_{10} + C_2^2 + D_1^2(x_0^2 - 1)] + D_1^2[A_3^2x_{11}^2 + 2A_3(B_3w + C_3)x_{11}] + B_3^2w^2 + 2B_3C_3w + C_3^2 = 0$$

Substituting (15), (16) and (22)-(25) into (6)-(10) and together with (17) we obtain the following six equations in the six unknowns $x_3, x_4, x_5, x_{10}, x_{11}, x_{12}$ and $w$.

$x_1 = (A_1x_{11} + B_3w + C_3)/D_2$, $x_2 = (A_1x_{10} + C_1)/D_1$, $x_4 = (A_2x_{10} + C_2)/D_1$, $x_5 = (A_4x_{11} + B_4w + C_4)/D_2$.

Substitution of the solutions for the above system into Eqs. (22)-(25), solution for $x_1, x_2, x_4, x_5, x_6$ can be obtained. By using the properties of homogeneous transformation matrix, $x_7, x_8$ and $x_9$ can be determined from

$$
\begin{bmatrix}
    x_7 \\
    x_8 \\
    x_9
\end{bmatrix} =
\begin{bmatrix}
    0 & -x_3 & x_2 \\
    x_3 & 0 & -x_1 \\
    -x_2 & x_1 & 0
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}.
$$
dependent and the system (5)-(13) has infinite number of solutions. Therefore, when \( \beta = \beta_0 \), the top platform has no definite position and orientation. Accordingly, when the two platforms are similar in the shapes, the mechanism becomes architecturally singular.

\[
\begin{array}{ccc}
  x_1 & x_2 & x_3 \\
  1 & -0.9622883400 & -0.193316013 &= -0.1913895459 \\
  2 & 0.9740800000 & -0.175610172 &= -0.1362892399 \\
  3 & -0.9622883400 & -0.193316013 &= 0.1913895459 \\
  4 & 0.9740800000 & -0.175610172 &= 0.1362892399 \\
  x_4 & x_5 & x_6 \\
  1 & 0.1749856216 & -0.9785627115 &= 0.1086059295 \\
  2 & 0.192903459 & 0.973387835 &= 0.1242703072 \\
  3 & 0.1749856216 & -0.9785627115 &= -0.1086059295 \\
  4 & 0.192903459 & 0.973387835 &= -0.1242703072 \\
  x_7 & x_8 & x_9 \\
  1 & -0.2082196960 & 0.071019981 &= 0.974869521 \\
  2 & 0.1108351131 & -0.1474256993 &= 0.9828438983 \\
  3 & 0.2082196960 & -0.071019981 &= 0.974869521 \\
  4 & -0.1108351131 & 0.1474256993 &= 0.9828438983 \\
  x_{10} & x_{11} & x_{12} \\
  1 & -0.0468437694 & -0.065026400 &= 3.38991711174 \\
  2 & -0.0355523947 & -0.076922495 &= 3.9218398082 \\
  3 & -0.0468437694 & -0.065026400 &= -3.38991711174 \\
  4 & -0.0355523947 & -0.076922495 &= -3.9218398082 \\
\end{array}
\]

Table 1: The 4 real solutions of the Crossed Type Stewart Platform

3 Homotopy Continuation Methods

Homotopy continuation methods can be used to solve polynomial systems numerically. For our model in the last section, let us denote the polynomial system (26)-(31) by \( P(q, z) = (p_1(q, z), ..., p_6(q, z)) = 0 \), \( z = (z_1, ..., z_6) \equiv (x_3, x_6, x_{10}, x_{11}, x_{12}, w) \in C^6 \), consider \( q = (R, R_0, \beta, \beta_0, L_1, ... , L_6) \in C^{10} \) as a parameter vector. First of all, the Bézout number [8] of our system is \( 2^6 = 64 \). Define a trivial start system

\[
Q(z) = \begin{bmatrix} z_1^2 - 1 \\ z_2^2 - 1 \\ \vdots \\ z_6^2 - 1 \end{bmatrix} = 0
\]  

and then follow the curves in the real variable \( t \) which make up the solution set of

\[
0 = H(z, t) = (1 - t)cQ(z) + tP(z)
\]  

where \( c \in C^1 \) is a randomly chosen parameter [9, 10]. These solution paths \( z(t) \) can be followed from the initial points, the solutions of \( H(z, 0) = Q(z) = 0 \), at \( t = 0 \) to all solutions of the original problem \( P(z) = 0 \) at \( t = 1 \) using standard numerical techniques.
Extensive numerical experiment with the homotopy (35) using the start system \( Q(z) = 0 \) in (34) to solve \( P(q, z) = 0 \) with different set of parameters shows that for generic \( q = (R, R_0, \beta, \beta_0, L_1, ..., L_6) \in C^{10}, P(q, z) = 0 \) has 28 solutions. Therefore \( P(q, z) = 0 \) is a deficient polynomial system [11]. When we fix a parameter \( q \) we have solved with 28 solutions, then for any given \( q \) we can solve \( P(q, z) = 0 \) by following only 28 paths. From our experience, the average CPU time to obtain all 28 solutions of \( P(q, z) = 0 \) is about 35 seconds on a Sparc Station ECL using double precision complex arithmetics. Furthermore, for a given platform, \( R, R_0, \beta \) and \( \beta_0 \) are fixed. Now, the parameter vector in our system \( P(q, z) = 0 \) becomes \( q = (L_1, ..., L_6) \in C^6 \). Follow the same procedure described above, the average CPU time to find all isolated solutions of \( P(q, z) = 0 \) is reduced to 6 seconds. Further, it follows from Remark 2.1 that the solution paths are in pairs. Therefore we need only to follow 14 of the solution paths.

4 Numerical Example

We select the following parameters for both the Non-crossed Type and Crossed Type Stewart Platform

\[
(R, R_0) = (1, 2), \quad (L_1, ..., L_6) = (4.3, 4.2, 4.3, 4.4, 4.5, 4.05).
\]

For Non-crossed Type Stewart Platform \((\beta, \beta_0) = (\pi/12, \pi/4)\), for Crossed Type Stewart Platform \((\beta, \beta_0) = (-\pi/12, \pi/4)\). Using the procedure described in the preceding sections we obtain 28 solutions for both Crossed Type Stewart Platform and Non-crossed Type Stewart Platform. The real solutions of the two platforms are listed in Table 1 and Table 2 respectively.

5 Conclusions

In this paper, we studied the forward displacement analysis of two classes of Stewart platform using one unified mathematical model. The direct kinematic equations can be reduced to solving a system of six second degree polynomial equations in six unknowns. It was shown that both of the two classes of Stewart platform have exactly 28 distinct, finite solutions and the solutions are in pairs. We need only to find 14 of the 28 solutions. Further, for any given platform an efficient nonlinear homotopy can be used to find the 14 solutions by following 14 continuation curves and the CPU time on a Sun Sparc Station ECL in double precision complex arithmetic is less than 3 seconds.

References


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Table 2: The 8 real solutions of the Non-crossed Type Stewart Platform