On a Solution of a Singular Integral Equation Appearing in the Diffraction Theory

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Abstract: This paper considers a singular integral equation which appears in the analysis of plane wave diffraction on a planar periodic metal strip grating. The kernel of the equation is the cotangent function which depends on the difference of the arguments. By transferring the equation into the complex domain a solution of the equation is found in a closed analytical form.

Key-Words: Plane wave diffraction, singular integral equation, residue techniques

1 Introduction
In the analysis of plane wave diffraction on a planar periodic metal strip grating one encounters a Hilbert-type singular integral equation whose kernel is the cotangent function which depends on the difference of the arguments. In [1], a solution of this equation is found in the closed form which contains some singular integrals. In the present paper by using the methods of the complex function theory a solution of the equation is found in a simple closed analytical form which contains no integrals.

2 Formulation of the Equation
The equation which is to be solved has the form

\[ \int_{-A}^{A} f(y) \cot \frac{x-y}{2} \, dy = e^{inx}, \quad n = 0, \pm 1, \pm 2, \ldots; -A < x < A \]  \hspace{0.5cm} (1)

where \( \varphi(y) \) is either \( \cos nx \) or \( \sin nx \). The unknown function \( F(y) \) should satisfy the same integral boundary condition i.e.

\[ \int_{-A}^{A} F(y) \, dy = 0 \]  \hspace{0.5cm} (4)

3 Solution of the Equation
By making a change of variables

\[ e^{iy} = z, \quad e^{ix} = z_0 \]  \hspace{0.5cm} (5)

eqn. (3) is transferred into the complex domain and takes on the form

\[ \int_{\Gamma} \left( \ln \frac{z_0 + z}{z_0 - z} \right) \frac{dz}{z} = \varphi \left( \frac{\ln z_0}{i} \right) \]  \hspace{0.5cm} (6)

where \( \Gamma \) is the arc of the unit circle \( |z| = 1 \) defined by \(-A \leq \arg z \leq A\). By virtue of (4) and (5) we have

\[ \int_{\Gamma} \left( \ln \frac{z}{z_0} \right) \frac{dz}{z} = i \int_{-A}^{A} F(y) \, dy = 0 \]  \hspace{0.5cm} (7)

so that eqn. (6) becomes

\[ \int_{\Gamma} \left( \ln \frac{z}{z_0} \right) \frac{dz}{z - z_0} = -\frac{1}{2} \varphi \left( \frac{\ln z_0}{i} \right) \]  \hspace{0.5cm} (8)
Eqn. (8) is a singular Cauchy-type integral equation whose solution is [2]

\[
F\left(\frac{\ln z_0}{i}\right) = \frac{1}{2\pi^2 R(z_0)} \int_{\Gamma} \frac{R(z)\phi\left(\frac{\ln z}{i}\right)}{z-z_0} \, dz + \frac{D}{R(z_0)} \tag{9}
\]

In eqn. (9) \(D\) is an unknown constant and

\[
R(z) = \sqrt{(z-e^{i\theta})(z-e^{-i\theta})} = \sqrt{z^2 - 2uz + 1} = \sum_{k=0}^{\infty} \rho_k(u)z^k, \quad |z| < 1
\]

where \(u = \cos\theta\) and the polynomials \(\rho(u)\) can be expressed in terms of the Legendre polynomials \(\rho_0(u) = 1, \rho_1(u) = 1, \rho_2(u) = -P_1(u)/P_0(u) = -4, \rho_{n+2}(u) = P_{n+2}(u) - uP_{n+1}\). By using (7) and (9) we have

\[
\int_{\Gamma} F\left(\frac{\ln z}{i}\right) \frac{dz}{z} = 0 = \int_{\Gamma} \frac{1}{2\pi^2 R(z)} \int_{\Gamma} \frac{R(w)\phi\left(\frac{\ln w}{i}\right)}{w-z} \, dw \frac{dz}{z} + D \int_{\Gamma} \frac{dz}{zR(z)}
\]

or after interchanging the order of integration

\[
\frac{1}{2\pi^2} \int_{\Gamma} R(w)\phi\left(\frac{\ln w}{i}\right) dw \int_{\Gamma} \frac{dz}{z(w-z)R(z)} + D \int_{\Gamma} \frac{dz}{zR(z)} = 0
\]

By substituting (11) into (9) we get a solution of eqn. (8)

\[
F\left(\frac{\ln z_0}{i}\right) = \frac{z_0}{2\pi^2 R(z_0)} \int_{\Gamma} \frac{R(z)\phi\left(\frac{\ln z}{i}\right)}{z-z_0} \, dz \tag{12}
\]

or by returning to the old variables \(x\) and \(y\)

\[
F(x) = \frac{1}{4\pi^2 \sqrt{\cos x - u}} \int_{-\alpha}^{\alpha} \frac{\cos y - u \varphi(y)dy}{\sin \frac{x-y}{2}} \tag{13}
\]

where the integral is understood in the sense of principal value. Finally, to obtain a solution of eqn. (1) it is necessary to put \(\varphi(y) = e^{iny}\) in eqn. (13) and perform the integration. This is done again by transferring the integral into the complex domain by means of (5) and by applying the residue techniques. Refering to [3] for details we arrive at the final form of a solution of eqn. (1) in the form

\[
f(x) = \frac{i}{2\pi\sqrt{\cos x - u}} \left\{ \sum_{k=0}^{n} \rho_{n-k}(u)e^{i(k+1/2)x}, \quad n \geq 1 \right. \\
\left. - \sum_{k=0}^{-n} \rho_{n+k}(u)e^{-i(k+1/2)x}, \quad n \leq -1 \right.
\]

4 Conclusion

In the paper we have considered a singular Hilbert-type integral equation which appears in the diffraction theory. A simple closed form of the solution of the equation is found. The method of the solution consisted in transferring the equation into the complex domain and using the residue techniques.

References: