Active RC Equivalent Structure of the Optimized Second-order PWL Dynamical System

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Abstract: - Active RC circuit model with a piecewise-linear (PWL) voltage-controlled voltage source (VCVS) and linear current-controlled voltage source (CCVS) for the second-order autonomous dynamical system realization is proposed.

Key-Words: Dynamical systems, second-order systems, state models, active PWL equivalent circuits

1 Introduction

State model of autonomous piecewise-linear (PWL) systems of Class $C$ [3],[4] can be expressed generally as

$$
\dot{x} = Ax + b h(w^T x), \quad A \in \mathbb{R}^{3 \times 3}, \quad x, b, w \in \mathbb{R}^3, \quad (1)
$$

where the elementary PWL feedback function (Fig. 1)

$$
h(w^T x) = \frac{1}{2} \left( \begin{array}{c}
w^T x + 1 \\
w^T x - 1 
\end{array} \right) \quad (2)
$$

contains the regions $D_0$ and $D_{+1}$ ($D_{-1}$). The dynamical behavior of the system is determined by two characteristic polynomials related to these individual regions [3], i.e.

$$
P(s) = (s - \mu_1)(s - \mu_2)(s - \mu_3) = 
\det (sI - A_0) = s^3 - p_1 s^2 + p_2 s - p_3, \quad (3)
$$

$$
Q(s) = (s - \nu_1)(s - \nu_2)(s - \nu_3) = 
\det (sI - A) = s^3 - q_1 s^2 + q_2 s - q_3, \quad (4)
$$

where $1$ is the unity matrix. Their roots represent the eigenvalues of the corresponding state matrices, which are mutually related by fundamental expression [5]

$$
A_0 = A + bw^T. \quad (5)
$$

Fig. 1: Simple memoryless PWL feedback function.

All systems of the Class $C$ having the same characteristic polynomials are qualitatively equivalent and they are related by linear topological conjugacy [4]. Typical systems of this class are the Chua’s model, both its canonical forms [3], and also recently derived optimized state model having the minimum sum of relative eigenvalue sensitivity squares with respect to the change of the individual state matrix parameters [7]. Just this low-sensitivity model is very useful as a prototype for practical chaotic system realization in the form of electronic circuit. It provides the possibility to utilize the block-decomposed form of the state matrix so that the design procedure can be started from the optimized second-order system and then extended by a simple way to the opti-
mized third-order case with upper block-triangular state matrix [7], [9]. The state matrix and the vectors in matrix equation (1) have the form

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
0 & 0 & a_{33}
\end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}
\]

Suppose one pair of the complex conjugate eigenvalues and one real eigenvalue in both outer and inner regions, (i.e. \( \nu_{1,2} = v' \pm j v'' \), \( \nu_3 \) - real; \( \mu_{1,2} = \mu' \pm j \mu'' \), \( \mu_3 \) - real). Then for the optimized low-sensitivity model the following conditions must be valid: [7]

\[
a_{11} = a_{22} = v', \quad a_{12} = -\left(v'\right)^2, \quad w_1 = w_3 = 1 \quad (7a)
\]

\[
a_{33} = v_3, \quad b_3 = \mu_3 - v_3, \quad w_2 = \frac{v''}{\mu'' - v''}K
\]

(9)

The optimization coefficient \( K \) is given as the real root of the quadratic equation [7]

\[
K^2 - 2K(M + 1) + 1 = 0, \quad \text{i.e.} \quad K = 1 + M \pm \sqrt{M(M + 2)},
\]

where the auxiliary parameter \( M \) is

\[
M = \frac{(\mu' - v')^2 + (\mu'' - v'')^2}{2\mu'' - v''} > 0, \quad (\mu', v'' \neq 0).
\]

The corresponding integrator-based block diagram has been derived for both second- and third-order cases [9] and also RLC active circuits, where, unlike the Chua’s model, the circuit parameters have direct relations to the model parameters, were introduced [11]. The intention of this contribution is to propose the corresponding active RC circuit model with PWL voltage-controlled voltage source.

2 Equivalent active RC circuit

Consider the second-order autonomous active RC circuit introduced in Fig. 2 containing linear current-controlled voltage source (CCVS) with transresistance \( W \) and voltage-controlled voltage source (VCVS) with PWL transfer characteristic function \( u_0 = f(W(i_1 + i_2)) \) having three segments with slopes \( A_0 \) and \( A_1 \) (Fig. 3) expressed as

\[
u_0 = A_0 W(i_1 + i_2) + (A_0 - A_1) h[W(i_1 + i_2)]
\]

![Fig. 2: Equivalent structure of the second-order autonomous subsystem with linear CCVS and piecewise-linear VCVS.](image-url)

where the currents are \( i_1 = \frac{u_1}{R_{01}}, \quad i_2 = \frac{u_2}{R_{02}} \). Choosing capacitor voltages \( u_1, u_2 \) as the state variables, the current Kirchhoff’s equations of this circuit can be written in the basic form

\[
C_1 \frac{d}{dt} i_1 = \frac{u_0 - u_1}{R_1} + \frac{W(i_1 + i_2) - u_1}{R_3} \quad (11a)
\]

\[
C_2 \frac{d}{dt} i_2 = \frac{u_0 - u_2}{R_2} + \frac{W(i_1 + i_2) - u_2}{R_4} \quad (11b)
\]

and then rewritten to the complete (non-normalized) state equation form, i.e.

\[
\begin{align}
\frac{d}{dt} u_1 &= \frac{(A_G + G_1)WG_0 - (G_1 + G_3 + G_0)}{C_1} u_1 + \frac{(A_G + G_3)WG_0}{C_1} u_2 + \frac{(A_0 - A_1)G_1}{C_1} h[i_1 + i_2] \quad (12a) \\
\frac{d}{dt} u_2 &= \frac{(A_G + G_1)WG_0 - (G_1 + G_3 + G_0)}{C_2} u_2 + \frac{(A_G + G_3)WG_0}{C_2} u_1 + \frac{(A_0 - A_1)G_2}{C_2} h[i_1 + i_2] \quad (12b)
\end{align}
\]
Utilizing the reference values of voltage $E$ (Fig. 3), resistance $R_0$, and capacitance $C_0$ the normalized state variables including the time scaling can be given as

$$\begin{align*}
x &= \frac{u_1}{E}, \\
y &= \frac{u_2}{E}, \\
\tau &= \frac{t}{RC_0}
\end{align*}$$

(13a,b,c)

Then the corresponding normalized capacitances and resistances are

$$\begin{align*}
\alpha &= \frac{C_1}{C_0}, \\
\beta &= \frac{C_2}{C_0}, \\
r_{01} &= \frac{1}{g_{01}} = \frac{R_{01}}{R_0}, \\
r_{02} &= \frac{1}{g_{02}} = \frac{R_{02}}{R_0}, \\
r_1 &= \frac{1}{g_1} = \frac{R_1}{R_1}, \\
r_2 &= \frac{1}{g_2} = \frac{R_2}{R_0}, \\
r_3 &= \frac{1}{g_3} = \frac{R_3}{R_0}, \\
r_4 &= \frac{1}{g_4} = \frac{R_4}{R_0}
\end{align*}$$

(14a,b)

(14c,d)

(14e,f)

(14g,h)

Denoting $k = \text{sgn}(R_0C_0)$ and choosing $W = R_0$ the state equations (12) can be rewritten into the normalized forms

$$\begin{align*}
\dot{x} &= \frac{dx}{d\tau} = \frac{k}{\alpha} \left[ (A_1g_1 + g_2g_{01} - (g_1 + g_2 + g_{01}))x + \frac{k}{\beta} (A_2g_2 + g_3g_{02} - (g_2 + g_3 + g_{02}))y \right], \\
\dot{y} &= \frac{dy}{d\tau} = \frac{k}{\beta} \left[ (A_1g_1 + g_2g_{01} - (g_1 + g_2 + g_{01}))x + \frac{k}{\alpha} (A_2g_2 + g_3g_{02} - (g_2 + g_3 + g_{02}))y \right].
\end{align*}$$

(15)

(16)

Comparing them with the general matrix form (6) for the second-order subsystem the following equations can be obtained:

$$\begin{align*}
a_{11} &= \frac{k}{\alpha} \left[ (A_1g_1 + g_2g_{01} - (g_1 + g_2 + g_{01})) \right], \\
b_1 &= \frac{k}{\alpha} (A_0 - A_1)g_1, \\
a_{12} &= \frac{k}{\alpha} (A_1g_1 + g_2g_{02} - (g_1 + g_2 + g_{02})), \\
b_2 &= \frac{k}{\alpha} (A_0 - A_1)g_2, \\
a_{22} &= \frac{k}{\beta} \left[ (A_2g_2 + g_3g_{02} - (g_2 + g_3 + g_{02})) \right], \\
b_3 &= \frac{k}{\beta} (A_0 - A_2)g_3, \\
a_{23} &= \frac{k}{\beta} (A_2g_2 + g_3g_{01} - (g_2 + g_3 + g_{01})), \\
b_4 &= \frac{k}{\beta} (A_0 - A_2)g_4.
\end{align*}$$

(17)

(18)

and then utilized as independent formulas for the design of the individual circuit parameters. For the case when $\alpha$, and $k$ are chosen as free parameters the results are summarized in the following design formulas, where the general state model (6) is considered:

\[ g_{01} = \frac{1}{r_{01}} = \frac{R_0}{R_{01}} = w_1, \quad g_{02} = \frac{1}{r_{02}} = \frac{R_0}{R_{02}} = w_2 \]

\[ g_1 = \frac{1}{r_1} = \frac{R_0}{R_1} = \frac{1}{1 - A_1} \left( \frac{\alpha a_{12} (w_{1} - a_{11})}{k w_2} \right) - w_1 \]

\[ g_2 = \frac{1}{r_2} = \frac{R_0}{R_2} = \frac{1}{1 - A_2} \left( \frac{\beta a_{21} (w_{2} - a_{22})}{k w_1} \right) - w_2 \]

\[ g_3 = \frac{1}{r_3} = \frac{R_0}{R_3} = \frac{\alpha a_{12}}{k w_1} - A_1 g_1 \]

\[ g_4 = \frac{1}{r_4} = \frac{R_0}{R_4} = \frac{\beta a_{21}}{k w_2} - A_2 g_2 \]

\[ A_0 = \frac{\alpha a_1}{k g_1} + A_1 \]

\[ A_0 = \frac{\beta a_2}{k g_2} + A_2 \]

The corresponding circuit model of the second-order subsystem can easily be developed from the structure shown in Fig. 2 and then used as the partial prototype for the practical realization of the opti-
mized third-order chaotic oscillator. Linear CCVS can be realized in the modified form with voltage inverter where both polarity of parameter $W$ are available. Such an arrangement gives us the possibility to change the sign of the individual circuit element parameters. Some other details of the realization conditions are presented in [10].

3 Conclusion

This contribution deals with the second-order nonlinear dynamical systems and their realizations using active RC circuit where linear CCVS and piecewise-linear VCVS with three-segment PWL symmetric transfer characteristic are utilized as the active elements. The dynamical behavior of such a system is determined by two sets of complex conjugate state matrix eigenvalues associated with the corresponding regions.

The contribution presents the complete and normalized state equations where the simple relation between model and circuit parameters entails also very simple design formulas in the synthesis procedure either in general form. The circuit proposed represents one possibility of the second-order system realization and can be easily extended also for the third-order system utilizing the block decomposition of the state matrix [7]. Such higher-order equivalent circuit can model also a chaotic behavior of the system.

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