Observers and Luenberger-type observers for 2D state-space models affected by unknown inputs

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Abstract:- In this paper, dead-beat unknown input observers (UIOs) for two-dimensional (2D) state-space models are investigated. Dead-beat UIOs are observers which produce an exact estimate of the original system state trajectory, after a finite number of evolution steps, independently of the system and observer initial conditions and of the inputs and the unknown disturbances that affect the system functioning. Necessary and sufficient conditions for the existence of dead-beat UIOs are provided. Comparisons with Luenberger-type UIOs are also carried on, and the extension of the paper results to the case of asymptotic UIOs with a given rate of convergence is finally discussed.

Key-Words:- Two-dimensional state-space models, dead-beat unknown input observers, Luenberger-type observers, asymptotic observers.

1. Introduction

The interest in two-dimensional (2D) systems goes back to the early seventies [8], and was initially motivated by the relevance of these models in seismology applications, X-ray image enhancement, image deblurring, digital picture processing, etc. More recently, many other interesting contexts where 2D systems prove to be the appropriate setting for carrying on a thorough and successful analysis have been enlightened.

Even though estimation problems for two-dimensional systems have been frequently afforded in very specific contexts, often related to image processing applications, the theoretical contributions on this subject are quite few [1, 2]. In particular, estimation problems for 2D systems affected by disturbances have not received sufficient attention, even though they represent a much more realistic and interesting problem to afford with respect to the standard "undisturbed" situation. Indeed, in the 1D context, there has been a long stream of research on this subject, which originated in the seventies and flourished in the eighties [11, 13], but still represents a very lively topic of research, mostly due to the extreme relevance of unknown input observers (UIOs) in contexts like fault detection, motion systems, etc. [7, 10].

In this paper dead-beat unknown input observers for 2D quarter-plane causal discrete state-space models, described by a Fornasini-Marchesini model [8], are introduced, and necessary and sufficient conditions for their existence are provided. As for standard dead-beat observers [1], it turns out that the problem can be efficiently solved by means of a polynomial approach, as the main focus turns out to be on the possible UIO transfer matrices, while the state-space models that realize them play no significant role, provided that they are finite memory.

The paper is organized as follows. In section 2, 2D systems and dead-beat unknown input observers are introduced, and necessary and sufficient conditions for the existence of a dead-beat UIO are given. Comparisons with Luenberger-type UIOs are carried on in section 3. Finally, in section 4, the problem extension to the case of asymptotic UIOs with rate of convergence greater than an assigned positive real number ρ is addressed.

2. 2D systems and observers

Consider the 2D (quarter plane causal) discrete state-space model described by the following equations [8]:

$$\begin{aligned} x(h+1,k+1) &= A_1 x(h,k+1) + A_2 x(h+1,k) \quad (1)\\ &+ B_1 u(h,k+1) + B_2 u(h+1,k) \\ &+ D_1 d(h,k+1) + D_2 d(h+1,k), \\ y(h,k) &= Cx(h,k) + Ju(h,k) + Kd(h,k), \quad (2) \end{aligned}$$

where the state, input, disturbance and output sequences $x(\cdot,\cdot)$, $u(\cdot,\cdot)$, $d(\cdot,\cdot)$ and $y(\cdot,\cdot)$ are defined on the discrete plane $\mathbb{Z}\times\mathbb{Z}$ and take values in $\mathbb{R}^{\mathtt{x}}$, $\mathbb{R}^{\mathtt{u}}$, $\mathbb{R}^{\mathtt{d}}$ and $\mathbb{R}^{\mathtt{y}}$, respectively. $A_1, A_2, B_1, B_2, D_1, D_2, C, J$ and K are real matrices of suitable dimensions. A 2D system of this type will be denoted by $\Sigma = (A_1, A_2, B_1, B_2, D_1, D_2, C, J, K)$. The initial conditions are assigned by specifying the local state values $x(i,-i)$, $i \in \mathbb{Z}$, namely by assigning the system *initial global state* $\mathcal{X}_0 := \{x(i,-i) : i \in \mathbb{Z}\}$. The input to output transfer matrix is $W_u(z_1,z_2) = C(I_{\mathtt{x}} - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2) + J$, meanwhile the disturbance to output transfer matrix is $W_d(z_1,z_2) = C(I_{\mathtt{x}} - A_1z_1 - A_2z_2)^{-1}(D_1z_1 + D_2z_2) + K$. A 2D observer [1, 2] for the 2D system (1)-(2) is a 2D system of the form

$$\begin{aligned} z(h+1,k+1) &= F_1z(h,k+1) + F_2z(h+1,k) \quad (3)\\ &+ G_1\begin{bmatrix} u(h,k+1)\\ y(h,k+1)\end{bmatrix} + G_2\begin{bmatrix} u(h+1,k)\\ y(h+1,k)\end{bmatrix}\\ \hat{x}(h,k) &= Hz(h,k) + R\begin{bmatrix} u(h,k)\\ y(h,k)\end{bmatrix}, \quad (4)\end{aligned}$$

having $u(\cdot,\cdot)$ and $y(\cdot,\cdot)$ as its inputs and the estimate $\hat{x}(\cdot,\cdot)$ of $x(\cdot,\cdot)$ as its output. We denote by $\hat{\Sigma} = (F_1,F_2,G_1,G_2,H,R)$ the observer, by $\hat{W}(z_1,z_2)$ the observer transfer matrix, namely

$$\begin{aligned} \hat{W}(z_1,z_2) &= [\,\hat{W}_u(z_1,z_2) \quad \hat{W}_y(z_1,z_2)\,]\\ &= H(I_{\mathtt{z}} - F_1z_1 - F_2z_2)^{-1}(G_1z_1 + G_2z_2) + R, \quad (5)\end{aligned}$$

$\mathtt{z}$ being the observer dimension, and by $\mathcal{Z}_0$ the observer initial global state. The sizes of $\hat{W}(z_1,z_2), \hat{W}_u(z_1,z_2)$ and $\hat{W}_y(z_1,z_2)$ are $\mathtt{x}\times(\mathtt{u}+\mathtt{y})$, $\mathtt{x}\times\mathtt{u}$ and $\mathtt{x}\times\mathtt{y}$, respectively.

Definition 1. A 2D observer (3)-(4) is said to be a *dead-beat unknown input observer (UIO)* if

- it is finite-memory ;
- the *estimate error* $e(h,k) := x(h,k) - \hat{x}(h,k)$ goes to zero, within a finite number of steps, i.e. $e(h,k) = 0$ for $h+k > N, \exists\, N$, for every choice of the initial global states $\mathcal{X}_0$ and $\mathcal{Z}_0$ and for every input sequence $u(h,k), h,k \in \mathbb{Z}, h+k \geq 0$ and every unknown input sequence $d(h,k), h,k \in \mathbb{Z}, h+k \geq 0$.

Remark: A 2D system is said to be *finite memory* if its free state evolution goes to zero within a finite number of steps, for every choice of its initial global state. It is well-known that finite memory systems realize finite impulse response (FIR) 2D filters, namely filters with a polynomial transfer matrix. Conversely, every FIR filter can be realized via a finite memory 2D system [3, 8].

In order to obtain necessary and sufficient conditions for the existence of a dead-beat unknown input observer, we first introduce the $(\mathtt{x}+\mathtt{y})\times\mathtt{x}$ *PBH (observability) matrix* [1, 2]

$$\mathcal{O}(z_1,z_2) := \begin{bmatrix} I_{\mathtt{x}} - A_1z_1 - A_2z_2\\ C\end{bmatrix}. \quad (6)$$

We let $\mathcal{V}(\mathcal{O})$ denote the variety of its maximal (i.e., $\mathtt{x}$th) order minors. Corresponding to the PBH matrix, we introduce the *Bézout equation* in the unknown polynomial matrices $Q(z_1,z_2)$ and $P(z_1,z_2)$, namely

$$I_{\mathtt{x}} = [\,Q(z_1,z_2) \quad P(z_1,z_2)\,]\begin{bmatrix} I_{\mathtt{x}} - A_1z_1 - A_2z_2\\ C\end{bmatrix}. \quad (7)$$

The Bézout equation (7) is solvable if and only the observability matrix is right zero prime or, equivalently $\mathcal{V}(\mathcal{O})$ is empty. When so, a complete parametrization of its solutions is available. Let $M^{-1}(z_1,z_2)N(z_1,z_2)$ be a left coprime matrix fraction description (MFD) [12] of the state to output transfer matrix $C(I_{\mathtt{x}} - A_1z_1 - A_2z_2)^{-1}$, so that

$$M^{-1}(z_1,z_2)N(z_1,z_2) = C(I_{\mathtt{x}} - A_1z_1 - A_2z_2)^{-1}. \quad (8)$$

This amounts to saying that the (left zero prime) 2D polynomial matrix $[\,N(z_1,z_2) \quad -M(z_1,z_2)\,] \in \mathbb{R}[z_1,z_2]^{\mathtt{y}\times(\mathtt{x}+\mathtt{y})}$ represents a minimal left annihilator of the PBH observability matrix. If $[\,\bar{Q}(z_1,z_2) \quad \bar{P}(z_1,z_2)\,]$ is any solution of the Bézout equation (7), the set of all solutions of (7) can be parametrized as follows

$$\begin{aligned} [\,Q(z_1,z_2) \quad P(z_1,z_2)\,] &= [\,\bar{Q}(z_1,z_2) \quad \bar{P}(z_1,z_2)\,] \quad (9)\\ &+ T(z_1,z_2)\,[\,N(z_1,z_2) \quad -M(z_1,z_2)\,],\end{aligned}$$

as T varies in $\mathbb{R}[z_1,z_2]^{\mathtt{x}\times\mathtt{y}}$.

We are in a position, now, to provide a characterization of 2D systems endowed with dead-beat UIOs. The characterization we will obtain will explicitly determine, as in [1], only the (polynomial) UIO transfer matrix $\hat{W}(z_1,z_2)$ and not the observer state equations. However, since every FIR 2D filter may be realized by means of a finite memory 2D system, as previously remarked, this result will create no problem at all. In other words, all the results of the paper will work independently of the choice of the specific realization of $\hat{W}(z_1,z_2)$, provided that it is finite memory. However, since every 2D polynomial transfer matrix admits an infinite number of finite memory realizations, the problem solution will never be unique.

Proposition 1. [4] Given a 2D state-space model (1)-(2), consider the disturbance to output system matrix

$$S_{d,y}(z_1,z_2) := \begin{bmatrix} I_{\mathtt{x}} - A_1z_1 - A_2z_2 & D_1z_1 + D_2z_2\\ C & -K\end{bmatrix}.$$

A necessary and sufficient condition for the existence of a dead-beat UIO is that there exists a polynomial pair $(Q(z_1,z_2), P(z_1,z_2))$ satisfying

$$[\,I_{\mathtt{x}} \quad 0\,] = [\,Q(z_1,z_2) \quad P(z_1,z_2)\,]\,S_{d,y}(z_1,z_2). \quad (10)$$

Remark: As it comes out of the previous proof [4], once we obtain a pair $(Q(z_1,z_2),\ P(z_1,z_2))$ satisfying (10), the corresponding dead-beat UIO exhibits the following polynomial transfer matrix

$$\begin{aligned} \hat{W}(z_1,z_2) &= [\,\hat{W}_u(z_1,z_2) \mid \hat{W}_y(z_1,z_2)\,] \quad (11)\\ &= [\,Q(z_1,z_2)(B_1z_1 + B_2z_2) - P(z_1,z_2)J \mid P(z_1,z_2)\,]\end{aligned}$$

and, conversely, any dead-beat UIO necessarily exhibits a transfer matrix $\hat{W}(z_1,z_2)$ that can be expressed as in (11), for some pair (Q,P) which solves (10).

In the following, we steadily assume that $\mathcal{O}(z_1, z_2)$ is right zero prime and devote our attention to determining necessary and sufficient conditions for the set $\mathcal{S} := \{[\, Q(z_1, z_2) \quad P(z_1, z_2)\,]$ satisfying (10)$\}$, to be not empty.

THEOREM 2. [4] Set $\bar{\mathtt{d}} := \operatorname{rank}\left(\begin{bmatrix} D_1 z_1 + D_2 z_2 \\ -K \end{bmatrix}\right)$ and let $[\, H_d(z_1, z_2) \quad H_k(z_1, z_2)\,]$ be an MLA of $\begin{bmatrix} D_1 z_1 + D_2 z_2 \\ -K \end{bmatrix}$. The following conditions are equivalent:

i) the set $\mathcal{S}$ is not empty, namely there exists a dead-beat UIO;

ii) $\Gamma_1(z_1, z_2) := H_d(z_1, z_2)(I_{\mathtt{x}} - A_1 z_1 - A_2 z_2) + H_k(z_1, z_2) C \in \mathbb{R}[z_1, z_2]^{(\mathtt{x}+\mathtt{y}-\bar{\mathtt{d}})\times \mathtt{x}}$ is a right zero prime matrix.

REMARKS 1) If $\Gamma_1(z_1, z_2)$ is right zero prime, and hence the corresponding Bézout equation

$$I_{\mathtt{x}} = X(z_1, z_2)\Gamma_1(z_1, z_2) \tag{12}$$

is solvable, then corresponding to any solution X, we get [4] both a pair $(Q, P) \in \mathcal{S}$ given by

$$[Q(z_1, z_2)|P(z_1, z_2)] = X(z_1, z_2)[H_d(z_1, z_2)|H_k(z_1, z_2)] \tag{13}$$

and the corresponding transfer matrix of a dead-beat UIO

$$\begin{aligned} \hat{W}(z_1, z_2) &= X(z_1, z_2)\,[\, H_d(z_1, z_2) \quad H_k(z_1, z_2)\,] \\ &\cdot \begin{bmatrix} B_1 z_1 + B_2 z_2 & 0 \\ -J & I_{\mathtt{y}} \end{bmatrix}. \end{aligned} \tag{14}$$

2) Condition v) in Theorem 2 shows that a necessary condition for the existence of an UIO is that $\mathtt{y} \geq \bar{\mathtt{d}}$. This result is consistent with the analogous result obtained for 1D state-space models [5, 6, 14].

We may now synthesize the previous results in the following algorithm for testing the existence and possibly constructing a dead-beat UIO:

1) Check whether $\Gamma_1(z_1, z_2)$ is right zero prime. If not, no dead-beat UIO can be obtained for the given system, otherwise go to the following step.

2) Find a solution $X(z_1, z_2)$ of the Bézout equation (12).

3) A pair (Q, P) belonging to $\mathcal{S}$ is given in (13), and the corresponding dead-beat UIO transfer matrix $\hat{W}(z_1, z_2)$ is given in (14).

4) Any finite memory realization of $\hat{W}(z_1, z_2)$, obtained, for instance, via the algorithm described in [8], provides the desired observer.

EXAMPLE 1 Consider the 2D system (1)-(2) with

$$A_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$D_1 = D_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ C = [\, 0 \quad 1\,],\ J = K = [\, 0\,].$$

We obtain $[\, H_d(z_1, z_2) \quad H_k(z_1, z_2)\,] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$ and $\Gamma_1(z_1, z_2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, which is unimodular (and therefore right zero prime). The only solution of (12) is $X(z_1, z_2) = \Gamma_1^{-1}(z_1, z_2) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, which leads to

$$\begin{aligned} Q(z_1, z_2) &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ P(z_1, z_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \\ \hat{W}(z_1, z_2) &= \begin{bmatrix} z_1 + z_2 & -1 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{15}$$

Among the infinitely many finite memory realizations of (15), a minimal one is

$$\begin{aligned} z(h+1, k+1) &= [\, 1 \quad 0\,] \begin{bmatrix} u(h, k+1) \\ y(h, k+1) \end{bmatrix} \\ &+ \quad [\, 1 \quad 0\,] \begin{bmatrix} u(h+1, k) \\ y(h+1, k) \end{bmatrix} \\ \hat{x}(h, k) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} z(h, k) + \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u(h, k) \\ y(h, k) \end{bmatrix}. \end{aligned}$$

Notice that this structure is the same one of a *reduced order* UIO, since from the structure of C we have that $x_2 = y$ needs not to be estimated, so that the previous output equation can be rewritten as

$$\hat{x}(h, k) = \begin{bmatrix} \hat{x}_1(h, k) \\ \hat{x_2}(h, k) \end{bmatrix} = \begin{bmatrix} z(h, k) - y(h, k) \\ y(h, k) \end{bmatrix}.$$

3. Luenberger observers

A 2D *Luenberger-type UIO* is described by the following equations

$$\begin{aligned} \hat{x}(h+1, k+1) &= (A_1 + L_1 C)\hat{x}(h, k+1) \\ &+ \quad (A_2 + L_2 C)\hat{x}(h+1, k) \\ &+ \ [\, B_1 + L_1 J \quad -L_1\,] \begin{bmatrix} u(h, k+1) \\ y(h, k+1) \end{bmatrix} \\ &+ \ [\, B_2 + L_2 J \quad -L_2\,] \begin{bmatrix} u(h+1, k) \\ y(h+1, k) \end{bmatrix}, \end{aligned} \tag{16}$$

where the symbols $\hat{x}, u$ and y take the usual meaning and all matrices involved have real entries. Notice that A_1, A_2, B_1, B_2, C and J are the same matrices appearing in the original system description (1)-(2). In other words, a Luenberger observer is a special case of observer (3)-(4) with these additional constraints:

- its dimension coincides with the system dimension, namely $\hat{\mathtt{x}} = \mathtt{x}$;
- the observer output coincides with the observer state;
- the observer matrices are related to the system matrices by means of the following conditions:

$$F_i = A_i + L_i C,\ G_i = [\, B_1 + L_i J \ \mid\ -L_i\,], i = 1, 2,$$
$$H = I_{\mathtt{x}},\ R = 0.$$

Such an observer is said to be a dead-beat UIO if it satisfies both conditions of Definition 1. The problem we now address is the following one: if a given 2D system admits a dead-beat UIO, and hence satisfies any of the equivalent conditions of Theorem 2, when among them there exists at least one of Luenberger-type?

As any dead-beat UIO must be a finite memory system, the polynomial matrix $I_{\mathtt{x}} - (A_1 + L_1C)z_1 - (A_2 + L_2C)z_2$ must be unimodular. Notice that the existence of a matrix pair (L_1, L_2) that makes $I_{\mathtt{x}} - (A_1 + L_1C)z_1 - (A_2 + L_2C)z_2$ unimodular (a necessary and sufficient condition for the existence of a Luenberger-type dead-beat observer, in case no unknown input affects the system [1]) is a more restrictive condition with respect to the right primeness of $\mathcal{O}(z_1, z_2)$. Indeed, it clearly corresponds to the existence of a solution (Q, P) of the Bézout equation (7) of the form $[\,Q(z_1, z_2) \quad P(z_1, z_2)\,] = [I_{\mathtt{x}} - (A_1 + L_1C)z_1 - (A_2 + L_2C)z_2]^{-1} \cdot [\,I_{\mathtt{x}} \quad -L_1z_1 - L_2z_2\,]$.

So, a necessary condition for the existence of Luenberger-type dead-beat UIO is that there exists a pair of matrices (L_1, L_2) such that $I_{\mathtt{x}} - (A_1 + L_1C)z_1 - (A_2 + L_2C)z_2$ is unimodular. Once the unimodularity condition is satisfied, however, the matrix pair (Q, P) thus obtained must satisfy the orthogonality condition

$$[\,Q(z_1, z_2) \quad P(z_1, z_2)\,] \begin{bmatrix} D_1z_1 + D_2z_2 \\ -K \end{bmatrix} = 0, \qquad (17)$$

namely $[\,I_{\mathtt{x}} \quad -L_1z_1 - L_2z_2\,] \begin{bmatrix} D_1z_1 + D_2z_2 \\ -K \end{bmatrix} = 0$. So, we have shown the following result.

Proposition 3. A necessary and sufficient condition for the existence of a Luenberger-type dead-beat UIO is that there exists a pair $(L_1, L_2) \in \mathbb{R}^{\mathtt{x}\times\mathtt{y}} \times \mathbb{R}^{\mathtt{x}\times\mathtt{y}}$ s.t.

$$D_i = -L_iK, \quad i = 1, 2, \qquad (18)$$

$$I_{\mathtt{x}} - (A_1 + L_1C)z_1 - (A_2 + L_2C)z_2 \text{ is unimodular.} \qquad (19)$$

It is immediately apparent that if any of the equivalent conditions of Theorem 2 holds, in general such a matrix pair (L_1, L_2) does not exists. For instance, if $K = 0$ condition (18) cannot be satisfied unless the system is unaffected by disturbances ($D_1 = D_2 = 0$).

Example 1 (*continued*) We have already proved the existence of a dead-beat UIO. A Luenberger-type dead-beat observer exists (corresponding to $L_1 = L_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ the polynomial matrix $I_2 - (A_1 + L_1C)z_1 - (A_2 + L_2C)z_2$ is unimodular). However, as $K = 0$, no Luenberger-type dead-beat UIO exists.

As a general statement, the existence of a dead-beat Luenberger-type UIO is a rare occurrence. However, when K is of full row rank and a dead-beat UIO exists, then a dead-beat Luenberger-type UIO can be found.

Proposition 4. Suppose that system (1)-(2) admits a dead-beat UIO, namely any of the equivalent conditions of Theorem 2 holds, and K is of full row rank. Then the dead-beat UIO transfer matrix $\hat{W}(z_1, z_2)$ is uniquely determined, and it can be implemented by means of a Luenberger-type observer (16).

Proof If K_r^{-1} denotes a right inverse of K, then $\mathtt{y} = \operatorname{rank}\left(\begin{bmatrix} (D_1z_1 + D_2z_2)K_r^{-1} \\ -I_{\mathtt{y}} \end{bmatrix}\right) \leq \operatorname{rank}\left(\begin{bmatrix} D_1z_1 + D_2z_2 \\ -K \end{bmatrix}\right) = \bar{\mathtt{d}}$. On the other hand, when a dead-beat UIO exists, $\mathtt{y} \geq \bar{\mathtt{d}}$ and hence it must be $\mathtt{y} = \bar{\mathtt{d}}$. So [4], the dead-beat UIO transfer matrix is uniquely determined and Γ_1 is square unimodular. Also, $H_d(z_1, z_2)$ is a square matrix. We now prove that it is unimodular, too. From

$$0 = [\,H_d(z_1, z_2) \quad H_k(z_1, z_2)\,] \begin{bmatrix} D_1z_1 + D_2z_2 \\ -K \end{bmatrix}$$

one easily gets $H_k(z_1, z_2) = H_d(z_1, z_2)(D_1z_1 + D_2z_2)K_r^{-1}$. So, $[H_d(z_1, z_2) \mid H_k(z_1, z_2)] = H_d(z_1, z_2) \cdot [I_{\mathtt{x}} \mid (D_1z_1 + D_2z_2)K_r^{-1}]$, and, since it is left factor prime, $H_d(z_1, z_2)$ must be unimodular. Also, condition $0 = [\,I_{\mathtt{x}} \quad (D_1z_1 + D_2z_2)K_r^{-1}\,] \begin{bmatrix} D_1z_1 + D_2z_2 \\ -K \end{bmatrix}$ ensures $D_1z_1 + D_2z_2 = (D_1z_1 + D_2z_2)K_r^{-1}K$, i.e.

$$D_i = D_iK_r^{-1}K, \qquad i = 1, 2. \qquad (20)$$

To prove our claim, it suffices to verifying that by assuming $L_i := -D_iK_r^{-1}, i = 1, 2$, both (18) and (19) are satisfied. By postmultiplying L_i by K and by using (20), condition (18) is immediately obtained for $i = 1, 2$. On the other hand $I_{\mathtt{x}} - (A_1 + L_1C)z_1 - (A_2 + L_2C)z_2 = I_{\mathtt{x}} - (A_1 - D_1K_r^{-1}C)z_1 - (A_2 - D_2K_r^{-1}C)z_2 = (I_{\mathtt{x}} - A_1z_1 - A_2z_2) + (D_1z_1 + D_2z_2)K_r^{-1}C = H_d^{-1}(z_1, z_2)[H_d(z_1, z_2)(I_{\mathtt{x}} - A_1z_1 - A_2z_2) + H_k(z_1, z_2)C] = H_d^{-1}(z_1, z_2)\Gamma_1(z_1, z_2)$. So, by the unimodularity of H_d and Γ_1, (19) holds.

4. Problem extensions

The dead-beat UIO design problem discussed in this paper can be easily generalized to the case of *asymptotic observers* with *rate of convergence* greater than an assigned positive real value $\rho \geq 1$ (see [2] for the standard asymptotic observers, in case no disturbances affect the system dynamics). Indeed, we may look for an UIO described as in (3)-(4) and satisfying the following conditions [2]:

- the variety of the *characteristic polynomial* $\Delta_{F_1,F_2}(z_1, z_2) := \det(I_{\mathtt{z}} - F_1z_1 - F_2z_2)$ does not intersect the closed polydisc of radius $\rho \geq 1$:

  $$\mathcal{P}_\rho := \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| \leq \rho \text{ and } |z_2| \leq \rho\};$$

- the estimate error $e(h, k) = x(h, k) - \hat{x}(h, k)$ asymptotically goes to zero, with rate of convergence greater than ρ^{-1}, i.e. there exists $\bar{\rho} > \rho$ such that $\lim_{h+k\to+\infty} \bar{\rho}^{h+k} \cdot e(h, k) = 0$, for every choice of the initial global states $\mathcal{X}_0$ and $\mathcal{Z}_0$ (provided that they are bounded) and for every input sequence $u(h, k), h, k \in \mathbb{Z}, h + k \geq 0$ and every unknown input sequence $d(h, k), h, k \in \mathbb{Z}, h + k \geq 0$.

REMARK: The first condition ensures that an *asymptotic UIO* is, in particular, asymptotically stable. Indeed, it is well known [9] that a 2D system is asymptotically stable if and only if its characteristic polynomial is devoid of zeros in $\mathcal{P}_1$. Moreover, if it is devoid of zeros in $\mathcal{P}_\rho$, with $\rho \geq 1$, we say that the system is *asymptotically stable with stability degree* ρ.

In order to solve this general problem we need to introduce some suitable algebra. Indeed, introduce [2] the set $\mathcal{H}_\rho := \{f(z_1, z_2) \in \mathbb{R}(z_1, z_2) : f \text{ is analytic in } \mathcal{P}_\rho\}$. Of course, $\mathbb{R}[z_1, z_2] = \lim_{\rho \to +\infty} \mathcal{H}_\rho$. For every $\rho > 0$ and even for $\rho \to +\infty$, $\mathcal{H}_\rho$ is a (unique factorization) domain. The units in $\mathcal{H}_\rho$ are those elements $f(z_1, z_2) \in \mathcal{H}_\rho$ whose inverses $f^{-1}(z_1, z_2)$ are, in turn, analytic in $\mathcal{P}_\rho$. If we denote by $\mathcal{V}(f)$ the variety of f, namely the set of zeros of f, then $f \in \mathcal{H}_\rho$ is a unit if and only if $\mathcal{V}(f) \cap \mathcal{P}_\rho = \emptyset$. Given $f_1, f_2, \dots, f_n$ in $\mathcal{H}_\rho$, the following facts are equivalent:

- the ideal generated by $f_1, f_2, \dots, f_n$ is $\mathcal{H}_\rho$;
- there exist $x_1, x_2, \dots, x_n \in \mathcal{H}_\rho$ such that $\sum_{i=1}^n x_i(z_1, z_2) f_i(z_1, z_2) = 1$;
- $\cap_{i=1}^n \mathcal{V}(f_i) \cap \mathcal{P}_\rho = \emptyset$.

Also, if we consider the set of matrices with entries in $\mathcal{H}_\rho$, it can be proved (by resorting to the previous equivalent facts or by suitably adjusting the same arguments provided in [2]) that given $M(z_1, z_2) \in \mathcal{H}_\rho^{p \times q}$ of full column rank the following facts are equivalent ones:

- $M(z_1, z_2)$ is right zero prime in $\mathcal{H}_\rho$, by this meaning that rank $M(z_1, z_2) = q$ for any $(z_1, z_2) \in \mathcal{P}_\rho$.
- the ideal generated by its maximal order minors is $\mathcal{H}_\rho$;
- the variety of the maximal order minors of M, $\mathcal{V}(M)$, does not intersect $\mathcal{P}_\rho$;
- the Bézout equation $X(z_1, z_2) M(z_1, z_2) = I_q$ is solvable in $\mathcal{H}_\rho$.

So, by simply moving from $\mathbb{R}[z_1, z_2]$ to $\mathcal{H}_\rho$, one may adapt all previous results to this case. All the Bézout equations involved in the dead-beat problem solution may be considered in the context of matrices with entries in $\mathcal{H}_\rho$. This way, we easily obtain necessary and sufficient conditions for the existence of an asymptotic UIO with rate of convergence greater than ρ. Indeed, such an UIO exists if and only if there exists a pair $(Q(z_1, z_2), P(z_1, z_2))$ with entries in $\mathcal{H}_\rho$ satisfying (10) or, equivalently, if and only if $\Gamma_1(z_1, z_2)$ is right zero prime in $\mathcal{H}_\rho$. If so, corresponding to any solution X (with entries in $\mathcal{H}_\rho$) of the Bézout equation $I_{\mathrm{x}} = X(z_1, z_2)\Gamma_1(z_1, z_2)$, we obtain the transfer matrix $\hat{W}(z_1, z_2)$, described in (14), of a possible asymptotic UIO. As proved in [3], $\hat{W}(z_1, z_2)$ can always be realized by means of an asymptotically stable 2D state-space model with stability degree ρ. Any such system provides the desired asymptotic UIO (with rate of convergence greater than ρ).

EXAMPLE 2 Consider the 2D system (1)-(2) with

$$A_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; A_2 = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$D_1 = -D_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; J = [\,0\,], \; K = [\,8\,].$$

The corresponding PBH matrix (6) is of full column rank in any point except for $(2, 2) \notin \mathcal{P}_1$. Indeed, $\mathcal{V}(\mathcal{O}) = \{(2, 2)\}$. One finds

$$[\, H_d(z_1, z_2) \quad H_k(z_1, z_2)\,] = \left[\begin{array}{cc|c} 8 & 0 & z_1 - z_2 \\ 0 & 8 & z_2 - z_1 \end{array}\right],$$
$$\Gamma_1(z_1, z_2) = \begin{bmatrix} 8 - 3z_1 - z_2 & z_1 - z_2 \\ z_2 - z_1 & 8 - z_1 - 3z_2 \end{bmatrix}.$$

As $\det(\Gamma_1) = 4(4 - z_1 - z_2)^2$, Γ_1 is unimodular in $\mathcal{H}_1$. So, an asymptotic UIO with rate of convergence greater than 1 (indeed, greater than any real number ρ satisfying $1 \leq \rho < 2$) exists and its transfer matrix is uniquely determined. In fact

$$X(z_1, z_2) = \frac{1}{4(4 - z_1 - z_2)^2} \begin{bmatrix} 8 - z_1 - 3z_2 & z_2 - z_1 \\ z_1 - z_2 & 8 - 3z_1 - z_2 \end{bmatrix},$$

and, correspondingly, we get the UIO transfer matrix

$$\hat{W}(z_1, z_2) = \frac{(z_1 - z_2)}{(4 - z_1 - z_2)^2} \begin{bmatrix} 0 & 2 - z_2 \\ 0 & z_1 - 2 \end{bmatrix}.$$

We aim at showing that such an asymptotic UIO transfer matrix may be realized by means of an asymptotically stable Luenberger-type UIO. To this end one may simply observe that K is trivially of full row rank. Therefore $\hat{W}$ is uniquely determined and it suffices to assume $L_1 = -D_1 K^{-1}$ and $L_2 = -D_2 K^{-1}$, namely $L_1 = \frac{1}{8}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -L_2$. A minimal asymptotically stable realization of $\hat{W}(z_1, z_2)$ that corresponds to a Luenberger-type UIO (with L_1 and L_2 previously given) is

$$F_1 = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, F_2 = \frac{1}{8}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}, G_1 = \frac{1}{8}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},$$
$$G_2 = \frac{1}{8}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \; H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

To conclude, we aim at providing an example which clearly shows that situations arise when all dead-beat UIO transfer matrices correspond to state-space realizations whose dimension is greater than the original system dimension. This, of course, prevents the existence of a Luenberger-type UIO.

EXAMPLE 3 Consider the 2D system (1)-(2) with

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$D_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; K = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

while B_1, B_2, D_1 and J are zero matrices. One gets

$$[\,H_d(z_1,z_2) \quad H_k(z_1,z_2)\,] = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & z_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

and hence

$$\Gamma_1(z_1,z_2) = \begin{bmatrix} 1 & z_2 & 0 & 0 \\ z_1 & 1 & z_2 & 0 \\ 0 & z_1 & 1 & z_2 \\ 0 & 0 & z_1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is a matter of computation to determine the possible polynomial matrices X which solve the Bézout equation $X(z_1,z_2)\Gamma_1(z_1,z_2) = I_{\mathrm{x}}$.

Correspondingly, we get

$$Q(z_1,z_2) = X(z_1,z_2)H_d(z_1,z_2)$$

$$\begin{bmatrix} 1+z_1z_2+2z_1^2z_2^2 & -z_2(1+2z_1z_2) & z_2^2(1-2z_1z_2) & 4z_2^3 \\ -z_1(1+2z_1z_2) & 1+2z_1z_2 & -z_2(1-2z_1z_2) & -4z_2^2 \\ z_1^2(1+2z_1z_2) & -z_1(1+2z_1z_2) & 1+z_1z_2-2z_1^2z_2^2 & 4z_1z_2^2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} p_1(z_1,z_2) \\ p_2(z_1,z_2) \\ p_3(z_1,z_2) \\ p_4(z_1,z_2) \end{bmatrix} [\,-z_1^3 \quad z_1^2 \quad -z_1(1-z_1z_2) \quad 1-2z_1z_2\,]$$

$$P(z_1,z_2) = X(z_1,z_2)H_k(z_1,z_2)$$

$$= \begin{bmatrix} z_2^3(1-2z_1z_2) & -z_2^3(5-2z_1z_2) \\ -z_2^2(1-2z_1z_2) & z_2^2(5-2z_1z_2) \\ z_2(1+z_1z_2-2z_1^2z_2^2) & -z_2(1+5z_1z_2-2z_1^2z_2^2) \\ 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} p_1(z_1,z_2) \\ p_2(z_1,z_2) \\ p_3(z_1,z_2) \\ p_4(z_1,z_2) \end{bmatrix} [\,-z_1z_2(1-z_1z_2) \quad -1+3z_1z_2-z_1^2z_2^2\,],$$

with $p_i(z_1,z_2)$ arbitrary in $\mathbb{R}[z_1,z_2]$. Moreover, $\hat{W}(z_1,z_2) = [\,0 \quad P(z_1,z_2)\,]$. As both entries in the first row of P have degree not smaller than 5, no state-space realization of dimension smaller than 5 can be found.

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