Abstract: In this paper, dead-beat unknown input observers (UIOs) for two-dimensional (2D) state-space models are investigated. Dead-beat UIOs are observers which produce an exact estimate of the original system state trajectory, after a finite number of evolution steps, independently of the system and observer initial conditions and of the inputs and the unknown disturbances that affect the system functioning. Necessary and sufficient conditions for the existence of dead-beat UIOs are provided. Comparisons with Luenberger-type UIOs are also carried on, and the extension of the paper results to the case of asymptotic UIOs with a given rate of convergence is finally discussed.

Key-Words: Two-dimensional state-space models, dead-beat unknown input observers, Luenberger-type observers, asymptotic observers.

1. Introduction

The interest in two-dimensional (2D) systems goes back to the early seventies [8], and was initially motivated by the relevance of these models in seismology applications, X-ray image enhancement, image deblurring, digital picture processing, etc. More recently, many other interesting contexts where 2D systems prove to be the appropriate setting for carrying on a thorough and successful analysis have been enlightened.

Even though estimation problems for two-dimensional systems have been frequently afforded in very specific contexts, often related to image processing applications, the theoretical contributions on this subject are quite few [1, 2]. In particular, estimation problems for 2D systems affected by disturbances have not received sufficient attention, even though they represent a much more realistic and interesting problem to afford with respect to the standard “undisturbed” situation. Indeed, in the 1D context, there has been a long stream of research on this subject, which originated in the seventies and flourished in the eighties [11, 13], but still represents a very lively topic of research, mostly due to the extreme relevance of unknown input observers (UIOs) in contexts like fault detection, motion systems, etc. [7, 10].

In this paper dead-beat unknown input observers for 2D quarter-plane causal discrete state-space models, described by a Fornasini-Marchesini model [8], are introduced, and necessary and sufficient conditions for their existence are provided. As for standard dead-beat observers [1], it turns out that the problem can be efficiently solved by means of a polynomial approach, as the main focus turns out to be on the possible UIO transfer matrices, while the state-space models that realize them play no significant role, provided that they are finite memory.

The paper is organized as follows. In section 2, 2D systems and dead-beat unknown input observers are introduced, and necessary and sufficient conditions for the existence of a dead-beat UIO are given. Comparisons with Luenberger-type UIOs are carried on in section 3. Finally, in section 4, the problem extension to the case of asymptotic UIOs with rate of convergence greater than an assigned positive real number $\rho$ is addressed.

2. 2D systems and observers

Consider the 2D (quarter plane causal) discrete state-space model described by the following equations [8]:

$$
x(h+1, k+1) = A_1 x(h, k+1) + A_2 x(h+1, k) + B_1 u(h, k+1) + B_2 u(h+1, k) + D_1 d(h, k+1) + D_2 d(h+1, k),
$$

$$
y(h, k) = C x(h, k) + J u(h, k) + K d(h, k),
$$

where the state, input, disturbance and output sequences $x(\cdot, \cdot)$, $u(\cdot, \cdot)$, $d(\cdot, \cdot)$ and $y(\cdot, \cdot)$ are defined on the discrete plane $\mathbb{Z} \times \mathbb{Z}$ and take values in $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^s$ and $\mathbb{R}^t$, respectively. $A_1, A_2, B_1, B_2, D_1, D_2, C, J$ and $K$ are real matrices of suitable dimensions. A 2D system of this type will be denoted by $\Sigma = (A_1, A_2, B_1, B_2, D_1, D_2, C, J, K)$. The initial conditions are assigned by specifying the local state values $x(i, -i)$, $i \in \mathbb{Z}$, namely by assigning the system initial global state $X_0 := \{x(i, -i) : i \in \mathbb{Z}\}$. The input to out-
put transfer matrix is \( W_u(z_1, z_2) = C(I_x - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2) + J \), meanwhile the disturbance to output transfer matrix is \( W_d(z_1, z_2) = C(I_x - A_1z_1 - A_2z_2)^{-1}(D_1z_1 + D_2z_2) + K \). A 2D observer [1, 2] for the 2D system (1)-(2) is a 2D system of the form

\[
z(h + 1, k + 1) = F_1z(h, k + 1) + F_2z(h + 1, k) + G_1u(h, k + 1) + G_2u(h + 1, k)
\]

\[
\hat{x}(h, k) = H z(h, k) + R [u(h, k) \ y(h, k)]^T
\]

having \( u(\cdot, \cdot) \) and \( y(\cdot, \cdot) \) as its inputs and the estimate \( \hat{x}(\cdot, \cdot) \) of \( x(\cdot, \cdot) \) as its output. We denote by \( \hat{\Sigma} = (F_1, F_2, G_1, G_2, H, R) \) the observer, by \( \hat{W}(z_1, z_2) \) the observer transfer matrix, namely

\[
\hat{W}(z_1, z_2) = \left[ \hat{W}_u(z_1, z_2) \quad \hat{W}_d(z_1, z_2) \right]
\]

\[
= H(I_x - F_1z_1 - F_2z_2)^{-1}(G_1z_1 + G_2z_2) + R.
\]

\[
\text{Remark: A 2D system is said to be finite memory if its free state evolution goes to zero within a finite number of steps, for every choice of its initial global state. It is well-known that finite memory systems realize finite impulse response (FIR) 2D filters, namely filters with a polynomial transfer matrix. Conversely, every FIR filter can be realized via a finite memory 2D system [3, 8].}

The Bézout equation (7) is solvable if and only the observability matrix is right zero prime or, equivalently \( V(\Omega) \) is empty. When so, a complete parametrization of its solutions is available. Let \( M^{-1}(z_1, z_2)N(z_1, z_2) \) be a left coprime matrix fraction description (MFD) [12] of the state to output transfer matrix \( C(I_x - A_1z_1 - A_2z_2)^{-1} \), so that

\[
M^{-1}(z_1, z_2)N(z_1, z_2) = C(I_x - A_1z_1 - A_2z_2)^{-1}.
\]

This amounts to saying that the (left zero prime) 2D polynomial matrix \( [N(z_1, z_2) - M(z_1, z_2)] \in \mathbb{R}[z_1, z_2]^{s_1 \times s_2} \) represents a minimal left annihilator of the PBH observability matrix. If \( [\hat{Q}(z_1, z_2) \quad P(z_1, z_2)] \) is any solution of the Bézout equation (7), the set of all solutions of (7) can be parametrized as follows

\[
\begin{bmatrix}
Q(z_1, z_2) \\
P(z_1, z_2)
\end{bmatrix}
\begin{bmatrix}
P(z_1, z_2)
\end{bmatrix}
\begin{bmatrix}
N(z_1, z_2) - M(z_1, z_2)
\end{bmatrix}
\]

as \( T \) varies in \( \mathbb{R}[z_1, z_2]^{s_1 \times s_2} \).

We are in a position, now, to provide a characterization of 2D systems endowed with dead-beat UIOs. The characterization we will obtain will explicitly determine, as in [1], only the (polynomial) UIO transfer matrix \( \hat{W}(z_1, z_2) \) and not the observer state equations. However, since every FIR 2D filter may be realized by means of a finite memory 2D system, as previously remarked, this result will create no problem at all. In other words, all the results of the paper will work independently of the choice of the specific realization of \( \hat{W}(z_1, z_2) \), provided that it is finite memory. However, since every 2D polynomial transfer matrix admits an infinite number of finite memory realizations, the problem solution will never be unique.

**Proposition 1.** [4] Given a 2D state-space model (1)-(2), consider the disturbance to output system matrix

\[
S_{d,g}(z_1, z_2) := \begin{bmatrix}
I_x & -A_1z_1 - A_2z_2 & D_1z_1 + D_2z_2
\end{bmatrix}
\]

A necessary and sufficient condition for the existence of a dead-beat UIO that is that there exists a polynomial pair \( (Q(z_1, z_2), P(z_1, z_2)) \) satisfying

\[
[I_x \ 0] = [Q(z_1, z_2) \ P(z_1, z_2)] S_{d,g}(z_1, z_2). \tag{10}
\]

**Remark:** As it comes out of the previous proof [4], once we obtain a pair \( (Q(z_1, z_2), P(z_1, z_2)) \) satisfying (10), the corresponding dead-beat UIO exhibits the following polynomial transfer matrix

\[
\hat{W}(z_1, z_2) = \begin{bmatrix}
W_u(z_1, z_2) & W_d(z_1, z_2)
\end{bmatrix}
\]

\[
= [Q(z_1, z_2)(B_1z_1 + B_2z_2) - P(z_1, z_2)J \ P(z_1, z_2)]
\]

and, conversely, any dead-beat UIO necessarily exhibits a transfer matrix \( \hat{W}(z_1, z_2) \) that can be expressed as in (11), for some pair \( (Q, P) \) which solves (10).
In the following, we steadily assume that \( \mathcal{O}(z_1, z_2) \) is right zero prime and devote our attention to determining necessary and sufficient conditions for the set \( S := \{ (Q(z_1, z_2), P(z_1, z_2)) \} \) satisfying (10), to be not empty.

**Theorem 2.** [4] Set \( \hat{\alpha} := \text{rank} \left( \begin{bmatrix} D_1 z_1 + D_2 z_2 \\ -K \end{bmatrix} \right) \) and let \( [H_d(z_1, z_2), H_k(z_1, z_2)] \) be an MLA of \( [D_1 z_1 + D_2 z_2, -K] \). The following conditions are equivalent:

i) the set \( S \) is not empty, namely there exists a deadbeat UIO;

\[ i) \Gamma_1(z_1, z_2) := H_d(z_1, z_2)(I_x - A_1 z_1 - A_2 z_2) + H_k(z_1, z_2)C \in \mathbb{R}[z_1, z_2]^{(s+\gamma-4) \times s} \text{ is a right zero prime matrix.} \]

**Remarks** 1) If \( \Gamma_1(z_1, z_2) \) is right zero prime, and hence the corresponding Bézout equation

\[ I_x = X(z_1, z_2) \Gamma_1(z_1, z_2) \tag{12} \]

is solvable, then corresponding to any solution \( X \), we get [4] both a pair \((Q, P) \in \mathbb{S}\) given by

\[ (Q(z_1, z_2), P(z_1, z_2)) = X(z_1, z_2) [H_d(z_1, z_2), H_k(z_1, z_2)] \tag{13} \]

and the corresponding transfer matrix of a dead-beat UIO

\[ \hat{W}(z_1, z_2) = X(z_1, z_2) [H_d(z_1, z_2), H_k(z_1, z_2)] \tag{14} \]

\[ \cdot \begin{bmatrix} B_1 z_1 + B_2 z_2 & 0 \\ -J & I_x \end{bmatrix} \]

2) Condition v) in Theorem 2 shows that a necessary condition for the existence of an UIO is that \( y \geq \hat{\alpha} \). This result is consistent with the analogous result obtained for 1D state-space models [5, 6, 14].

We may now synthesize the previous results in the following algorithm for testing the existence and possibly constructing a dead-beat UIO:

1) Check whether \( \Gamma_1(z_1, z_2) \) is right zero prime. If not, no dead-beat UIO can be obtained for the given system, otherwise go to the following step.

2) Find a solution \( X(z_1, z_2) \) of the Bézout equation (12).

3) A pair \((Q, P)\) belonging to \( S \) is given in (13), and the corresponding dead-beat UIO transfer matrix \( \hat{W}(z_1, z_2) \) is given in (14).

4) Any finite memory realization of \( \hat{W}(z_1, z_2) \) obtained, for instance, via the algorithm described in [8], provides the desired observer.

**Example 1** Consider the 2D system (1)-(2) with

\[ A_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]

\[ D_1 = D_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = [0 \ 1], \quad J = K = [0 \ 0]. \]

We obtain \[ [H_d(z_1, z_2), H_k(z_1, z_2)] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

and \( \Gamma_1(z_1, z_2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \), which is unimodular (and therefore right zero prime). The only solution of (12) is \( X(z_1, z_2) = \Gamma_1^{-1}(z_1, z_2) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \), which leads to

\[ Q(z_1, z_2) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad P(z_1, z_2) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \]

\[ \hat{W}(z_1, z_2) = \begin{bmatrix} z_1 + z_2 & -1 \\ 0 & 1 \end{bmatrix}. \tag{15} \]

Among the infinitely many finite memory realizations of (15), a minimal one is

\[ z(h+1,k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u(h,k+1) \\ y(h,k+1) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u(h,k+1) \\ y(h,k+1) \end{bmatrix} \]

\[ \hat{x}(h,k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} z(h,k) + \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u(h,k) \\ y(h,k) \end{bmatrix}. \]

Notice that this structure is the same one of a reduced order UIO, since from the structure of \( C \) we have that \( z_2 = y \) needs not to be estimated, so that the previous output equation can be rewritten as

\[ \hat{x}(h,k) = \begin{bmatrix} \hat{x}_1(h,k) \\ \hat{x}_2(h,k) \end{bmatrix} = \begin{bmatrix} z(h,k) - y(h,k) \\ y(h,k) \end{bmatrix}. \]

**3. Luenberger observers**

A 2D Luenberger-type UIO is described by the following equations

\[ \hat{x}(h+1,k+1) = (A_1 + L_1 C) \hat{x}(h,k+1) \]

\[ + (A_2 + L_2 C) \hat{x}(h+1,k) \]

\[ + [B_1 + L_1 J - L_1] \begin{bmatrix} u(h,k+1) \\ y(h,k+1) \end{bmatrix} \]

\[ + [B_2 + L_2 J - L_2] \begin{bmatrix} u(h+1,k) \\ y(h+1,k) \end{bmatrix}, \]

where the symbols \( \hat{x}, u \) and \( y \) take the usual meaning and all matrices involved have real entries. Notice that \( A_1, A_2, B_1, B_2, C \) and \( J \) are the same matrices appearing in the original system description (1)-(2). In other words, a Luenberger observer is a special case of observer (3)-(4) with these additional constraints:

- its dimension coincides with the system dimension, namely \( \hat{x} = x \);
- the observer output coincides with the observer state;
- the observer matrices are related to the system matrices by means of the following conditions:

\[ F_i = A_i + L_i C, \ G_i = [B_1 + L_1 J - L_1], \ i = 1, 2, \]

\[ H = I_x, \ R = 0. \]

Such an observer is said to be a dead-beat UIO if it satisfies both conditions of Definition 1. The problem we
now address is the following one: if a given 2D system
admits a dead-beat UIO, and hence satisfies any of the
equivalent conditions of Theorem 2, when among them
there exists at least one of Luenberger-type?

As any dead-beat UIO must be a finite memory sys-
tem, the polynomial matrix \( I_k - (A_1 + L_1 C) z_1 - (A_2 +
L_2 C) z_2 \) must be unimodular. Notice that the exis-
tence of a matrix pair \( (L_1, L_2) \) that makes \( I_k - (A_1 +
L_1 C) z_1 - (A_2 + L_2 C) z_2 \) unimodular (a necessary and
sufficient condition for the existence of a Luenberger-
type dead-beat observer, in case no unknown input
affects the system [1]) is a more restrictive condition
with respect to the right primeness of \( \mathcal{O}(z_1, z_2) \). In-
deed, it clearly corresponds to the existence of a so-
lution \((Q, P)\) of the Bézout equation (7) of the form
\[
\begin{bmatrix} Q(z_1, z_2) & P(z_1, z_2) \end{bmatrix} = \begin{bmatrix} I_k & (A_1 + L_1 C) z_1 - (A_2 + L_2 C) z_2 \end{bmatrix}^{-1},
\]

namely \( I_k - L_1 z_1 - L_2 z_2 \). Thus obtained must satisfy the orthogonality condition
\[
[Q(z_1, z_2) P(z_1, z_2)]\begin{bmatrix} D_1 z_1 + D_2 z_2 \\
\end{bmatrix} = 0, \tag{17}
\]

one easily gets \( H_{d}(z_1, z_2) = H_{d}(z_1, z_2) (D_1 z_1 +
D_2 z_2) K_r^{-1} \). So, \( \| H_{d}(z_1, z_2) \| \) such that \( H_{d}(z_1, z_2) \) is unimodular (a necessary and
sufficient condition for the existence of a dead-beat UIO exists, \( \gamma \geq \hat{\gamma} \) and hence it must be
\( \gamma = \hat{\gamma} \). So [4], the dead-beat UIO transfer matrix
is uniquely determined and \( \Gamma_1 \) is square unimodular.
Also, \( H_{d}(z_1, z_2) \) is a square matrix. We now prove that
it is unimodular, too. From
\[
o = \begin{bmatrix} H_{d}(z_1, z_2) & H_{d}(z_1, z_2) \end{bmatrix} \begin{bmatrix} D_1 z_1 + D_2 z_2 \\
\end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}, \tag{18}
\]

To prove our claim, it suffices to verifying that by as-
suming \( L_i := -D_i K_r^{-1}, i = 1, 2 \), both (18) and (19)
are satisfied. By postmultiplying \( L_i \) by \( K_1 \) and by us-
(20) condition (18) is immediately obtained for \( i = 1, 2 \). On the other hand \( I_k - (A_1 + L_1 C) z_1 - (A_2 +
L_2 C) z_2 = I_k - (A_1 - D_1 K_r^{-1} C) z_1 - (A_2 - D_2 K_r^{-1} C) z_2 =
(I_k - A_1 z_1 - A_2 z_2) + (D_1 z_1 + D_2 z_2) K_r^{-1} C =
H_{d}^{-1}(z_1, z_2) H_{d}(z_1, z_2)(I_k - A_1 z_1 - A_2 z_2) + H_{d}(z_1, z_2) C =
H_{d}^{-1}(z_1, z_2) \Gamma_1(z_1, z_2) \). So, by the unimodularity of \( H_{d} \)
and \( \Gamma_1 \), (19) holds.

4. Problem extensions

The dead-beat UIO design problem discussed in this pa-
cer can be easily generalized to the case of asymptotic
observers with rate of convergence greater than an
assigned positive real value \( \rho \geq 1 \) (see [2] for the standard
asymptotic observers, in case no disturbances affect the
system dynamics). Indeed, we may look for an UIO
described as in (3)-(4) and satisfying the following con-
ditions [2]:

- the variety of the characteristic polynomial
  \( \Delta_{F_1 F_2}(z_1, z_2) := \det(I_k - F_1 z_1 - F_2 z_2) \) does not
  intersect the closed polydisc of radius \( \rho \geq 1 \):
  \( P_\rho := \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq \rho \) and \( |z_2| \leq \rho \} ;

- the estimate error \( e(h, k) = x(h, k) - \hat{x}(h, k) \)
asymptotically goes to zero, with rate of convergence
  greater than \( \rho^{-1} \), i.e. there exists \( \rho > 0 \) such that
  \( \lim_{h \to +\infty} \rho^{h+k} e(h, k) = 0 \), for every choice of
  the initial global states \( X_0 \) and \( Z_0 \) (provided that
  they are bounded) and for every input sequence
  \( u(h, k), h, k \in \mathbb{Z}, h + k \geq 0 \) and every unknown
  input sequence \( d(h, k), h, k \in \mathbb{Z}, h + k \geq 0 \).
The following facts are equivalent:

tries in may be considered in the context of matrices with en-
adapt all previous results to this case. All the Bézout
Of course, the ideal generated by
and even for \( \rho \to +\infty \), \( H_\rho \) is a (unique factorization) domain. The units in \( H_\rho \) are those elements \( f(z_1, z_2) \in H_\rho \) whose inverses \( f^{-1}(z_1, z_2) \) are, in turn, analytic in \( P_\rho \). If we denote by \( V(f) \) the variety of \( f \), namely the set of zeros of \( f \), then \( f \in H_\rho \) is a unit if and only if \( V(f) \cap P_\rho = \emptyset \). Given \( f_1, f_2, \ldots, f_n \) in \( H_\rho \), the following facts are equivalent:

- the ideal generated by \( f_1, f_2, \ldots, f_n \) is \( H_\rho \);
- there exist \( x_1, x_2, \ldots, x_n \in H_\rho \) such that \( \sum_{i=1}^{\infty} f_i(z_1, z_2)x_i(z_1, z_2) = 1 \);
- \( \bigcap_{i=1}^{\infty} V(f_i) \cap P_\rho = \emptyset \).

Also, if we consider the set of matrices with entries in \( H_\rho \), it can be proved (by resorting to the previous equivalent facts or by suitably adjusting the same arguments provided in [2]) that given \( M(z_1, z_2) \in H_\rho^{p \times q} \) of full column rank the following facts are equivalent ones:

- \( M(z_1, z_2) \) is right zero prime in \( H_\rho \), by this meaning that rank \( M(z_1, z_2) = q \) for any \( (z_1, z_2) \in P_\rho \);
- the ideal generated by its maximal order minors is \( H_\rho \);
- the variety of the maximal order minors of \( M \), \( V(M) \), does not intersect \( P_\rho \);
- the Bézout equation \( X(z_1, z_2)M(z_1, z_2) = I_q \) is solvable in \( H_\rho \).

So, by simply moving from \( \mathbb{R}[z_1, z_2] \) to \( H_\rho \), one may adapt all previous results to this case. All the Bézout equations involved in the dead-beat problem solution may be considered in the context of matrices with entries in \( H_\rho \). This way, we easily obtain necessary and sufficient conditions for the existence of an asymptotic UIO with rate of convergence greater than \( \rho \). Indeed, such an UIO exists if and only if there exists a pair \( (Q(z_1, z_2), P(z_1, z_2)) \) with entries in \( H_\rho \) satisfying (10) or, equivalently, if and only if \( \Gamma_1(z_1, z_2) \) is right zero prime in \( H_\rho \). If so, corresponding to any solution \( X \) (with entries in \( H_\rho \)) of the Bézout equation \( L_\rho X = X(z_1, z_2)\Gamma(z_1, z_2) \), we obtain the transfer matrix \( \hat{W}(z_1, z_2) \), described in (14), of a possible asymptotic UIO. As proved in [3], \( \hat{W}(z_1, z_2) \) can always be realized by means of an asymptotically stable 2D state-space model with stability degree \( \rho \). Any such system provides the desired asymptotic UIO (with rate of convergence greater than \( \rho \)).

Example 2 Consider the 2D system (1)-(2) with

\[
A_1 = \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

\[
D_1 = -D_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = [1 \ 1], \quad J = [0], \quad K = [8].
\]

The corresponding PBH matrix (6) is of full column rank in any point except for \( (2, 2) \notin P_1 \). Indeed, \( V(\Omega) = \{(2, 2)\} \). One finds

\[
\begin{align*}
H_\rho(z_1, z_2) &= H_\rho(z_1, z_2) = \begin{bmatrix} 8 & 0 & z_1 - z_2 \\ 0 & 8 & z_2 - z_1 \end{bmatrix}, \\
\Gamma_1(z_1, z_2) &= \begin{bmatrix} 8 - 3z_1 - z_2 & z_1 - z_2 \\ z_2 - z_1 & 8 - 3z_1 - z_2 \end{bmatrix}.
\end{align*}
\]

As \( \det(\Gamma_1) = 4(4 - z_1 - z_2)^2 \), \( \Gamma_1 \) is unimodular in \( H_1 \). So, an asymptotic UIO with rate of convergence greater than 1 (indeed, greater than any real number \( \rho \) satisfying \( 1 \leq \rho < 2 \)) exists and its transfer matrix is uniquely determined. In fact

\[
X(z_1, z_2) = \frac{1}{4(4 - z_1 - z_2)^2} \begin{bmatrix} 8 - 3z_1 - 2z_2 & z_1 - z_2 \\ z_2 - z_1 & 8 - 3z_1 - 2z_2 \end{bmatrix},
\]

and, correspondingly, we get the UIO transfer matrix

\[
\hat{W}(z_1, z_2) = \frac{(z_1 - z_2)}{(4 - z_1 - z_2)^2} \begin{bmatrix} 0 & 2 - z_2 \\ 0 & z_1 - 2 \end{bmatrix}.
\]

We aim at showing that such an asymptotic UIO transfer matrix may be realized by means of an asymptotically stable Luenberger-type UIO. To this end one may simply observe that \( K \) is trivially of full row rank. Therefore \( \hat{W} \) is uniquely determined and it suffices to assume \( L_1 = -D_1K^{-1} \) and \( L_2 = -D_2K^{-1} \), namely

\[
L_1 = \frac{1}{8} \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad L_2 = \frac{1}{8} \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

To conclude, we aim at providing an example which clearly shows that situations arise when all dead-beat UIO transfer matrices correspond to state-space realizations whose dimension is greater than the original system dimension. This, of course, prevents the existence of a Luenberger-type UIO.

Example 3 Consider the 2D system (1)-(2) with

\[
A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad K = [1 \ 0] .
\]
while $B_1, B_2, D_1$ and $J$ are zero matrices. One gets

$$[H_d(z_1, z_2), H_k(z_1, z_2)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & z_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and hence

$$\Gamma_1(z_1, z_2) = \begin{bmatrix} 1 & z_2 & 0 & 0 \\ z_1 & 1 & z_2 & 0 \\ 0 & z_1 & 1 & z_2 \\ 0 & 0 & z_1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  

It is a matter of computation to determine the possible polynomial matrices $X$ which solve the Bézout equation $X(z_1, z_2) \Gamma_1(z_1, z_2) = I_k$.

Correspondingly, we get

$$Q(z_1, z_2) = X(z_1, z_2) H_d(z_1, z_2)$$

$$+ \begin{bmatrix} p_1(z_1, z_2) \\ p_2(z_1, z_2) \\ p_3(z_1, z_2) \\ p_4(z_1, z_2) \end{bmatrix} \begin{bmatrix} -z_1^3 & z_1^3 & -z_1(1-z_1 z_2) & 1-2z_1 z_2 \end{bmatrix}$$

$$P(z_1, z_2) = X(z_1, z_2) H_k(z_1, z_2)$$

$$= \begin{bmatrix} z_2^3(1-2z_1 z_2) & -z_2^3(5-2z_1 z_2) \\ -z_2^3(1-2z_1 z_2) & z_2^3(5-2z_1 z_2) \\ z_2(1+z_1 z_2-2z_1 z_2^2) & -z_2(1+5z_1 z_2-2z_1 z_2^2) \\ 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} p_1(z_1, z_2) \\ p_2(z_1, z_2) \\ p_3(z_1, z_2) \\ p_4(z_1, z_2) \end{bmatrix} \begin{bmatrix} -z_1 z_2(1-z_1 z_2) & -1+3z_1 z_2-z_1^2 z_2^2 \end{bmatrix},$$

with $p_k(z_1, z_2)$ arbitrary in $\mathbb{R}[z_1, z_2]$. Moreover, $\tilde{W}(z_1, z_2) = [0 \ P(z_1, z_2)]$. As both entries in the first row of $P$ have degree not smaller than 5, no state-space realization of dimension smaller than 5 can be found.

References: