On the Generalized Second–Order 2D System–Model

GEORGE E. ANTONIOU*, MARINOS T. MICHAEL
Image Processing and Systems Laboratory
Department of Computer Science
Montclair State University
Montclair, N.J 07043
USA
{george.antoniou, marinos.michael}@montclair.edu

Abstract : In this paper a generalized or singular two dimensional system–model of second order, $\mathbf{E}x(i_1+2, i_2+2) = \mathbf{A}_0 x(i_1+1, i_2+1) + \mathbf{A}_1 x(i_1+1, i_2) + \mathbf{A}_2 x(i_1, i_2+1)$, is introduced. Using this model a technique for computing its transfer function, using the discrete Fourier transform, is given. The algorithm is straight forward and has been implemented using the software package Matlab^{TM}. One step–by–step example, illustrating the application of the algorithm, is presented.

Keywords : 2D systems, Generalized systems, Singular systems, Transfer function, DFT(FFT)

1 Introduction

During the past two decades there has been extensive research in two dimensional (2D) systems. This is due to the extensive range of applications, especially in engineering and computing [1]-[3].

First- order generalized systems are applied in engineering as well as in biological and economic systems [4]–[7]. Second- order generalized systems, applied to power systems, were studied by Campbell and Rose [8]. These systems were also used in conjuction with the analysis and modelling of flexible beams [9]. Recently second–order generalized systems were used in circuit theory, linear control systems, filtering, mechanical system modeling and applied mathematics [10]–[12].

2D systems can be represented with a transfer function in polynomial form or with state space models [3]. State space based techniques play a very crucial role in the analysis and synthesis of 2D systems. An important problem is to determine the coefficients of a transfer function from its state space representation and vice versa. Leverrier–Fadeeva, discrete Fourier transform (DFT) algorithms and Vanderlmode matrices can be modified to be used for various models. The DFT has been used for the evaluation of the transfer function coefficients for linear, singular, multidimensional and regular second– order 2D systems [13]–[17].

In this paper a computer implementable algorithm is proposed for the computation of the transfer function for a new generalized 2D system–model that is also of second–order. The proposed algorithm determines the coefficients of the determinantal polynomial and the coefficients of the adjoint polynomial matrix, using the DFT. The computational speed of the method can be improved by using fast Fourier transform techniques.

2 *G2O2D* System

A generalized second–order two–dimensional (G2O2D) system–model has the following structure [18]:

$$
\begin{aligned}
\mathbf{E}x(i_1+2, i_2+2) &= \mathbf{A}_0 x(i_1+1, i_2+1) \\
&\quad +\mathbf{A}_1 x(i_1+1, i_2) \\
&\quad +\mathbf{A}_2 x(i_1, i_2+1) \\
&\quad +\mathbf{B}_1 u(i_1+1, i_2) \\
&\quad +\mathbf{B}_2 u(i_1, i_2+1) \qquad (1) \\
y(i_1, i_2) &= \mathbf{C}\, x(i_1, i_2)
\end{aligned}
$$

where, $x(i_1, i_2) \in \mathcal{R}^{\lambda}$, $u(i_1, i_2) \in \mathcal{R}^m$, $y(i_1, i_2) \in \mathcal{R}^p$; $\mathbf{A}_k$, for $k = 0, 1, 2$ and $\mathbf{E}, \mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{C}$, are real matrices of appropriate dimensions. Matrix $\mathbf{E}$ may be singular.

It is noted that this particular G2O2D system– model (1) is an extension of the 2D Fornasini–Marchesini model [19] to cover singular systems of second–order. For more 2D second-order structures the reader can refer to [18].

*Supported by the 2003 Student Faculty Award and the Margaret and Herman Sokol Faculty Award.

Applying the 2D $z_i,\ i = 1, 2$, transform to system–model (1) , with zero initial conditions, the transfer function is found to be:

$$
\begin{aligned}
T(z_1, z_2) &= \mathbf{C}\left[\mathbf{E}z_1^2 z_2^2 - \mathbf{A}_0 z_1 z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2\right]^{-1} \\
&\quad \cdot(\mathbf{B}_1 z_1 + \mathbf{B}_2 z_2) \qquad (2)
\end{aligned}
$$

In the following section an interpolative approach is developed for determining the transfer function $T(z_1, z_2)$, given the matrices $\mathbf{A}_k,\ k = 0, 1, 2$ and $\mathbf{B}_1,\ \mathbf{B}_2,\ \mathbf{C}$ using the 2D DFT. For the sake of completeness a brief description of the 2D DFT follows.

3 2D Discrete Fourier Transform

Consider the finite sequences $X(k_1, k_2)$ and $\tilde{X}(r_1, r_2)$, $k_i, r_i = 0, \cdots, M_i, i = 1, 2$. In order for the sequences $X(k_1, k_2)$ and $\tilde{X}(r_1, r_2)$, to constitute a 2D DFT pair the following relations should hold [20]:

$$
\begin{aligned}
\tilde{X}(r_1, r_2) &= \sum_{k_1=0}^{M_1}\sum_{k_2=0}^{M_2} X(k_1, k_2) W_1^{-k_1 r_1} W_2^{-k_2 r_2} \qquad (3) \\
X(k_1, k_2) &= \frac{1}{R}\sum_{r_1=0}^{M_1}\sum_{r_2=0}^{M_2} \tilde{X}(r_1, r_2) W_1^{k_1 r_1} W_2^{k_2 r_2} \qquad (4)
\end{aligned}
$$

where,

$$R = (M_1 + 1)(M_2 + 1) \qquad (5)$$

$$W_i = e^{(2\pi j)/(M_i+1)},\ i = 1, 2 \qquad (6)$$

$X,\ \tilde{X}$ are discrete argument matrix valued functions, with dimensions $p \times m$.

In the following section an interpolative approach is developed for determining the transfer function $T(s)$, given the matrices $\mathbf{A}_i, i = 1, 2, \mathbf{E}, \mathbf{B}_1,\ \mathbf{B}_2,\ \mathbf{C}$, using the 2D DFT.

4 Algorithm

The transfer function of the G2O2D system–model (1) is,

$$\mathbf{T}(z_1, z_2) = \frac{\mathbf{N}(z_1, z_2)}{d(z_1, z_2)} \qquad (7)$$

where,

$$
\begin{aligned}
\mathbf{N}(z_1, z_2) &= \mathbf{C}\ \mathtt{adj}\left[\mathbf{E}z_1^2 z_2^2 - \mathbf{A}_0 z_1 z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2\right] \\
&\quad \cdot[\mathbf{B}_1 z_1 + \mathbf{B}_2 z_2] \qquad (8) \\
d(z_1, z_2) &= \mathtt{det}\left[\mathbf{E}z_1^2 z_2^2 - \mathbf{A}_0 z_1 z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2\right] (9)
\end{aligned}
$$

Equations (8) and (9) can be written in polynomial form as follows:

$$\mathbf{N}(z_1, z_2) = \sum_{\lambda_1=0}^{n_{max}^P}\sum_{\lambda_2=0}^{n_{max}^P} \mathbf{P}_{\lambda_1,\lambda_2} z_1^{\lambda_1} z_2^{\lambda_2} \qquad (10)$$

with, $n_{max}^P := max(2\lambda - 1)$. The numerator coefficients $\mathbf{P}_{\lambda_1,\lambda_2}$ are matrices with dimensions $(p \times m)$.

$$d(z_1, z_2) = \sum_{\lambda_1=0}^{n_{max}^q}\sum_{\lambda_2=0}^{n_{max}^q} q_{\lambda_1,\lambda_2} z_1^{\lambda_1} z_2^{\lambda_2} \qquad (11)$$

where, $n_{max}^q := max(2\lambda - 1)$. The denominator coefficients q_{λ_1,λ_2} are scalars.

The numerator polynomial matrix $\mathbf{N}(z_1, z_2)$ and the denominator polynomial $d(z_1, z_2)$ can be numerically computed at $R = (r + 1)^2$, points, equally spaced on the unit 2D disc. The R points are chosen as $(z_1, z_2) = [v(i_1), v(i_2)],\ i_1, i_2 = 0, \cdots, r$, with $r = 2\lambda - 1$, according to definition as:

$$v_1(i) = v_2(i) = W^{-i},\ \ \forall\ \ i = 0, \ldots, r. \qquad (12)$$

where,

$$W_i = e^{(2\pi j)/(r+1)},\ \ i = 1, 2 \qquad (13)$$

The values of the transfer function (7) at the R points are the corresponding 2D DFT coefficients.

4.1 Denominator Polynomial

To evaluate the denominator coefficients q_{λ_1,λ_2}, define,

$$
\begin{aligned}
a_{i_1,i_2} &= \mathtt{det}\left[\mathbf{E}v_1^2(i_1)v_2^2(i_2) - \mathbf{A}_0 v_1(i_1)v_2(i_2)\right. \\
&\quad \left. -\mathbf{A}_1 v_1(i_1) - \mathbf{A}_2 v_2(i_2)\right] \qquad (14)
\end{aligned}
$$

Therefore using equations (11) and (14), a_{i_1,i_2} can be defined as,

$$a_{i_1,i_2} = d[v_1(i_1), v_2(i_2)] \qquad (15)$$

Provided that at least one of $a_{i_1,i_2} \neq 0$.

Equations (11), (12) and (15) yield

$$a_{i_1,i_2} = \sum_{\lambda_1=0}^{r}\sum_{\lambda_2=0}^{r} q_{\lambda_1,\lambda_2} W^{-(i_1\lambda_1 + i_2\lambda_2)} \qquad (16)$$

In the above equation (16) it is obvious that $[a_{i_1,i_2}]$ and $[q_{\lambda_1,\lambda_2}]$ form a 2D DFT pair. Therefore the coefficients $[q_{\lambda_1,\lambda_2}]$ can be computed using the inverse 2D DFT, as follows:

$$q_{\lambda_1,\lambda_2} = \frac{1}{R}\sum_{i_1=0}^{r}\sum_{i_2=0}^{r} a_{i_1,i_2} W^{(i_1\lambda_1 + i_2\lambda_2)} \qquad (17)$$

4.2 Numerator Polynomial

To evaluate the numerator matrix polynomial $\mathbf{P}_{\lambda_1,\lambda_2}$, define

$$\begin{aligned}\mathbf{F}_{i_1,i_2} &= \mathbf{C}\ \mathtt{adj}\ [\mathbf{E}v_1^2(i_1)v_2^2(i_2) \\ &\quad -\mathbf{A}_0 v_1(i_1)v_2(i_2) - \mathbf{A}_1 v_1(i_1) - \mathbf{A}_2 v_2(i_2)] \\ &\quad \cdot[\mathbf{B}_1 v_1(i_1) + \mathbf{B}_2 v_2(i_2)] \qquad (18)\end{aligned}$$

Using equations (10) and (18), $\mathbf{F}_{i_1,i_2}$ can be defined as,

$$\mathbf{F}_{i_1,i_2} = \mathbf{N}[v_1(i_1), v_2(i_2)] \qquad (19)$$

Equations (10), (12) and (19) yield

$$\mathbf{F}_{i_1,i_2} = \sum_{\lambda_1=0}^{r}\sum_{\lambda_2=0}^{r} \mathbf{P}_{\lambda_1,\lambda_2} W^{-(i_1\lambda_1+i_2\lambda_2)} \qquad (20)$$

In the above equation (20), $[\mathbf{F}_{i_1,i_2}]$, $[\mathbf{P}_{\lambda_1,\lambda_2}]$ form a 2D DFT pair. Therefore the coefficients $\mathbf{P}_{\lambda_1,\lambda_2}$ can be computed, using the inverse 2D DFT, as follows:

$$\mathbf{P}_{\lambda_1,\lambda_2} = \frac{1}{R}\sum_{i_1=0}^{r}\sum_{i_2=0}^{r} \mathbf{F}_{i_1,i_2} W^{(i_1\lambda_1+i_2\lambda_2)} \qquad (21)$$

A salient example, simple yet illustrative of the theoretical concepts presented in this work, follow below:

5 Numerical Example

Consider the following G2O2D, two–input two–output system– model

$$\begin{aligned}\mathbf{E}x(i_1+2,i_2+2) &= \mathbf{A}_0 x(i_1+1,i_2+1) \\ &\quad +\mathbf{A}_1 x(i_1+1,i_2) \\ &\quad +\mathbf{A}_2 x(i_1,i_2+1) \\ &\quad +\mathbf{B}_1 u(i_1+1,i_2) \\ &\quad +\mathbf{B}_2 u(i_1,i_2+1) \qquad (22) \\ y(i_1,i_2) &= \mathbf{C}\ x(i_1,i_2)\end{aligned}$$

where,

$$\begin{aligned}\mathbf{E} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{A}_0 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \\ \mathbf{A}_1 &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\ \mathbf{B}_1 &= \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \\ \mathbf{C} &= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\end{aligned}$$

Since $\lambda = 2$, the $r = 2\cdot\lambda - 1 = 3$. Therefore $R = (r+1)^2 = 16$. The direct application of the proposed algorithm yields:

$$\mathbf{F}_{00} = \begin{bmatrix} 1.0000 & 4.0000 \\ -45.0000 & 2.0000 \end{bmatrix}$$

$$\mathbf{F}_{01} = \begin{bmatrix} -1.7639 - 3.8042j & 0.1180 - 2.2654j \\ -14.7984 + 30.1563j & -9.9721 - 1.9879j \end{bmatrix}$$

$$\mathbf{F}_{02} = \begin{bmatrix} -6.2361 - 2.3511j & -2.1180 - 2.7144j \\ 9.7984 + 10.7516j & -1.0279 + 5.3431j \end{bmatrix}$$

$$\mathbf{F}_{03} = \begin{bmatrix} -6.2361 + 2.3511j & -2.1180 + 2.7144j \\ 9.7984 - 10.7516j & -1.0279 - 5.3431j \end{bmatrix}$$

$$\mathbf{F}_{10} = \begin{bmatrix} 5.7361 - 2.7144j & 0.8090 - 2.4899j \\ 16.4443 + 25.8828J & 4.6631 - 2.0409j \end{bmatrix}$$

$$\mathbf{F}_{11} = \begin{bmatrix} -6.2361 + 1.1756j & -1.4271 - 2.9389j \\ 30.9787 + 7.0534j & 0.1910 + 5.2901j \end{bmatrix}$$

$$\mathbf{F}_{12} = \begin{bmatrix} 5.8992 + 9.8208j & -3.9271 + 3.3022j \\ 16.2812 - 16.5312j & 8.5451 + 3.3022j \end{bmatrix}$$

$$\mathbf{F}_{13} = \begin{bmatrix} 5.7361 - 2.9919j & 2.3541 + 1.5388j \\ -11.9721 - 5.8981j & -0.5000 - 3.4410j \end{bmatrix}$$

$$\mathbf{F}_{20} = \begin{bmatrix} 1.2639 + 2.2654j & -0.3090 - 0.2245j \\ -1.4443 - 14.2331j & -3.1631 - 5.2043j \end{bmatrix}$$

$$\mathbf{F}_{21} = \begin{bmatrix} 1.2639 - 5.7921j & -4.3541 - 0.3633j \\ -3.0279 - 18.1028j & -0.5000 - 0.8123j \end{bmatrix}$$

$$\mathbf{F}_{22} = \begin{bmatrix} -1.7639 - 1.9021j & 1.9271 + 4.7553j \\ -15.9787 - 11.4127j & 1.3090 - 8.5595j \end{bmatrix}$$

$$\mathbf{F}_{23} = \begin{bmatrix} 1.0000 - 5.7063j & 3.3090 - 0.9511j \\ -2.7639 + 2.9717j & -6.7812 - 1.2286j \end{bmatrix}$$

$$\mathbf{F}_{30} = \begin{bmatrix} 1.2639 - 2.2654j & -0.3090 + 0.2245j \\ -1.4443 + 14.2331j & -3.1631 + 5.2043j \end{bmatrix}$$

$$\mathbf{F}_{31} = \begin{bmatrix} -6.3992 + 3.1307j & -0.5729 + 3.2164j \\ 6.2188 + 11.5312j & 2.9549 + 3.2164j \end{bmatrix}$$

$$\mathbf{F}_{32} = \begin{bmatrix} 1.0000 + 5.7063j & 3.3090 + 0.9511j \\ -2.7639 - 2.9717j & -6.7812 + 1.2286j \end{bmatrix}$$

$$\mathbf{F}_{33} = \begin{bmatrix} -1.7639 + 1.9021j & 1.9271 - 4.7553j \\ -15.9787 + 11.4127j & 1.3090 + 8.5595j \end{bmatrix}$$

Using (21), the numerator matrix polynomials are,

$$\begin{aligned}
P_{00} &= P_{01} = 0, \mathbf{P}_{02} = \begin{bmatrix} 3 & 1 \\ -3 & 1 \end{bmatrix}, P_{03} = 0 \\
P_{10} &= 0, \mathbf{P}_{11} = \begin{bmatrix} 1 & 3 \\ -13 & -1 \end{bmatrix}, \mathbf{P}_{12} = \begin{bmatrix} 0 & -1 \\ 0 & 5 \end{bmatrix} \\
P_{13} &= 0 \\
\mathbf{P}_{20} &= \begin{bmatrix} -3 & 0 \\ -11 & -4 \end{bmatrix}, \mathbf{P}_{21} = \begin{bmatrix} 3 & 0 \\ -15 & 0 \end{bmatrix}, P_{22} = 0 \\
\mathbf{P}_{23} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \\
\mathbf{P}_{30} &= \mathbf{P}_{31} = 0, \mathbf{P}_{32} = \begin{bmatrix} -3 & 0 \\ -3 & 0 \end{bmatrix}, P_{33} = 0
\end{aligned}$$

Once the denominator and the adjoint matrix coefficients have been computed, the transfer function $\mathbf{T}(z_1, z_2)$ is determined as,

$$\mathbf{T}(z_1, z_2) = \frac{\mathbf{z} \begin{bmatrix} 0 & 0 & \mathbf{P}_{02} & 0 \\ 0 & \mathbf{P}_{11} & \mathbf{P}_{12} & 0 \\ \mathbf{P}_{20} & \mathbf{P}_{21} & 0 & \mathbf{P}_{23} \\ 0 & 0 & \mathbf{P}_{32} & 0 \end{bmatrix} \mathbf{z}'}{\mathbf{z} \begin{bmatrix} 0 & 0 & q_{02} & 0 \\ 0 & q_{11} & q_{12} & 0 \\ q_{20} & q_{21} & 0 & q_{23} \\ 0 & 0 & q_{32} & 0 \end{bmatrix} \mathbf{z}'}$$

where,

$$\mathbf{z} = \begin{bmatrix} 1 & z_1 & z_1^2 & z_1^3 \end{bmatrix}$$

or

$$\mathbf{T}(z_1, z_2) = \frac{\begin{aligned} &\begin{bmatrix} 3 & 1 \\ -3 & 1 \end{bmatrix} z_2^2 + \begin{bmatrix} 1 & 3 \\ -13 & -1 \end{bmatrix} z_1 z_2 \\ &+ \begin{bmatrix} 0 & -1 \\ 0 & 5 \end{bmatrix} z_1 z_2^2 + \begin{bmatrix} -3 & 0 \\ -11 & -4 \end{bmatrix} z_1^2 \\ &+ \begin{bmatrix} 3 & 0 \\ -15 & 0 \end{bmatrix} z_1^2 z_2 + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} z_1^2 z_2^3 \\ &+ \begin{bmatrix} -3 & 0 \\ -3 & 0 \end{bmatrix} z_1^3 z_2^2 \end{aligned}}{-z_2^2 - 2z_1 z_2 - 2z_1 z_2^2 - z_1^2 + z_1^2 z_2 - z_1^2 z_2^3 - z_1^3 z_2^2}$$

Finally the transfer function of our two–input two–output GSO2D system–model is,

$$\mathbf{T}(z_1, z_2) = \frac{\begin{bmatrix} \alpha_{11}(z_1, z_2) & : & \alpha_{12}(z_1, z_2) \\ \ldots\ldots\ldots & : & \ldots\ldots\ldots \\ \alpha_{21}(z_1, z_2) & : & \alpha_{22}(z_1, z_2) \end{bmatrix}}{-z_2^2 - 2z_1 z_2 - 2z_1 z_2^2 - z_1^2 + z_1^2 z_2 - z_1^2 z_2^3 - z_1^3 z_2^2} \quad (23)$$

where,

$$\begin{aligned}
\alpha_{11}(z_1, z_2) &= +3z_2^2 + z_1 z_2 - 3z_1^2 + 3z_1^2 z_2 - 3z_1^3 z_2^2 \\
\alpha_{12}(z_1, z_2) &= +z_2^2 + 3z_1 z_2 - z_1 z_2^2 + z_1^2 z_2^3 \\
\alpha_{21}(z_1, z_2) &= -3z_2^2 - 13z_1 z_2 - 11z_1^2 - 15z_1^2 z_2 - 3z_1^3 z_2^2 \\
\alpha_{22}(z_1, z_2) &= +z_2^2 - z_1 z_2 + 5z_1 z_2^2 - 4z_1^2 + z_1^2
\end{aligned}$$

The correctness of the transfer function (23) can easily be verified using (2).

6 Complexity of the Algorithm

The proposed algorithm has two parts. In the first part the matrices $\mathbf{F}_{i_1,i_2}$ and the scalars a_{i_1,i_2} are evaluated with a cost of $pmR\lambda^3$ operations. In the second part the coefficients of $\mathbf{P}_{\lambda_1,\lambda_2}$ and q_{λ_1,λ_2} are evaluated using the DFT with a cost of $pmR^2 + R^2$ operations. For more efficient computation, especially for high order systems, fast Fourier methods can be used to implement the DFT [20].

Due to the inherent modularity and the algorithmic structure of the presented method high parallelism is permitted. In this case the computation of each determinant a_{i_1,i_2}, (16), and each matrix product $\mathbf{F}_{i_1,i_2}$, (20), can be distributed over a number of processing elements, considerably reducing the computation time of the algorithm.

7 Conclusions

An algorithm was presented for the computation of the transfer function for a new generalized second–order 2D system– model . The technique is using the DFT algorithm and has been implemented with the software package Matlab^{TM}. To further improve the computational speed of the algorithm, fast Fourier transform (FFT) techniques and FPGA (Field–Programmable Gate Array) based machines can be used.

References

[1] T. Kaczorek, *Positive 1D and 2D Systems*, Communication and Control Engineering Series, Berlin: Springer–Verlag, 2001.

[2] N.K. Bose, *Applied Multidimensional Systems Theory*, New York: Van Nostrand Reinhold, 1982.

[3] Wu–Sheng Lu and A. Antoniou, *Two–Dimensional Digital Filters*, New York: Marcel–Dekker, 1992.

[4] S.L. Campbell, Singular Systems of Differential Equations, London: Pittman, 1982.

[5] F.L. Lewis, "A survey of linear singular systems", *Circuits Systems and Signal Processing*, Vol. 5, No. 1, p.p. 3-36, 1986.

[6] L. Dai, Singular Control Systems, *Lecture Notes in Control and Information Sciences*, Berlin: Springer-Verlag, 1989.

[7] D.G. Luenberger, and Arbel,A., "Singular dynamic Leondieff systems ", *Econometrica*, Vol. 45, p. 991, 1977.

[8] S.L. Campbell, and N.J. Rose.,"A second order singular linear system arising in electric power analysis",*Int. J. Systems Science*, Vol. 13, pp. 101-108, 1982.

[9] W. Marszalek, and H. Unbehauen., "Second order generalized linear systems arising in analysis of flexible beams", *Proceedings of the 31st Conference on Decision and Control*, pp. 3514–3518, Arizona, 1992.

[10] D. Henrion, M. Sebek and V. Kucera, "Robust pole placement for design second–order systems: an LMI approach", IFAC Symposium on Robust Control, Milan, Italy, 2003.

[11] Y.Chahlaoui, D.Lemonnier, K. Meerbergen, A. Vandendorpe and P. Van Dooren, "Model reduction of second order systems", MTNS2002, Fifteenth International Symposium on Mathematical Theory of Networks and Systems, University of Notre Dame, 2002.

[12] C. Goodwin, *Real time block recursive parameter estimation of second order systems*, Ph.D Thesis, Nottingham Trent University, U.K, 1997.

[13] T. Lee, "A simple method to determine the characteristic function $f(s) = |s\mathbf{I} - \mathbf{A}|$ by discrete Fourier series and fast Fourier transform", IEEE Trans. Circuit Syst., Vol. CAS-23, p. 242, 1976.

[14] L.E. Paccagnella, *et al.*, "FFT calculation of a determinantal polynomial", IEEE Trans. Autom. Control, Vol. AC-21, p. 401, 1976.

[15] G.E. Antoniou, Glentis, G.O.A., Varoufakis, S.J., and Karras, D.A.,"Transfer function determination of singular systems using the DFT", IEEE Trans. Circuit Syst., Vol. CAS-36, pp. 1140-1142, 1989.

[16] K.S.Yeung, and Kumbi, F., "Symbolic matrix inversion with application to electronic circuits", IEEE Trans. Circuit Syst., Vol. CAS-35, No. 2, pp. 235-239, 1988.

[17] G.E. Antoniou, M.T. Michael, "Computing the transfer function for Second-order 2D Systems", IEEE International Symposium on Circuits and Systems, Vancouver, Canada, ISCAS'2004.

[18] G.E. Antoniou, "Second–order 2D state space systems: Models and transfer functions", WSEAS International Conference on Circuits, Systems and Computers, Corfu, Greece, CCSC'2003.

[19] E. Fornasini, E. Marchesini, "Doubly indexed dynamical systems: state space models and structural properties", Math. System Theory, Vol. 12, No. 1, pp. 59-72, 1978.

[20] S.K. Mitra, *Digital Signal Processing*, McGraw–Hill, New York, 2000.