Performance Analysis of Statistical Multiplexers with Finite Number of Input Links and a Train Arrival Process

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Abstract: - In this paper, we present an exact discrete-time queuing analysis of a statistical multiplexer with a finite number of input links and whose arrival process is correlated and consists of a train of a fixed number of fixed-length packets. The functional equation describing this queuing model is manipulated and transformed into a mathematical tractable form. This allows us to apply the final value theorem to extract an exact expression for the steady-state probability generating functions (pgfs) of the queue length and packet arrivals. From the pgf of the queue length, several performance measures such as probabilities of buffer overflow, mean buffer occupancy and mean packet delay are derived. The transform approach used in the present analysis provides a general framework under which similar types of functional equations, arising in the performance analysis of statistical multiplexers, can be tackled. One of the salient characteristics of the analysis presented here is that it does not rely on matrix-geometric concepts such as spectral decomposition, probability generating matrix, and left and right eigenvectors

Key-Words: - ATM multiplexers**,** Performance analysis, Transform analysis, Train arrivals, Functional equations, Discrete-time queues.

1 Introduction

In this paper, we consider a statistical multiplexer with N input links, having the same transmission rate, and one output link. The multiplexer is assumed to have an infinite buffer capacity and packets are served on a FCFS basis. The arrival process to this multiplexer is correlated and consists of a fixed-length packet-train arrival process. The main thrust behind our interest in investigating the impact of the above train arrival process on the performance of switching elements stems from the fact that such train arrival models are often encountered in the performance evaluation of large-scale ATM switching networks. For example, in some ATM environments, large external data frames (e.g voice or IP frames) are segmented at the edge of an ATM network into fixed-length ATM cells (mini-cells). In other applications [1], the edge ATM components are synchronized on a slot basis (time to carry one ATM cell) while the backbone (interior) ATM components operate on a mini-slot basis (a smaller time unit, used to carry a mini-cell). In this special environment, the edge ATM switching elements convert each ATM cell into a fixed number of m mini-cells which are then switched downstream the network. In this sense, the train arrival process considered in this paper captures the mini-cell arrival process on each internal link. Discrete-time queuing models with correlated train arrivals are also encountered in various other applications whereby customers are messages (eg. Frames or jumbo packets) composed of multiple fixed-length packets, see eg. [2].

In this paper, we model an ATM multiplexer as a discrete-time queue, whose arrival process consists of mini-cell arrivals (thereafter referred to train arrivals). A functional equation describing this system has been derived in [1]. However, as pointed out in [1], it is very difficult to derive the exact probability generating function (pgf) of the buffer occupancy from the functional equation. As a result, we manipulated and transformed the functional equation describing this queuing model into a mathematical tractable form. This allows us to apply the final value theorem to extract an exact expression for the corresponding steady-state probability generating function. The proposed transform approach for solving this functional equation is an extension of an earlier approach [3] in the analysis of ATM multiplexers with correlated arrivals. From this pgf, we derive a closed-form expression for the mean buffer occupancy and show that it is equivalent to the expression derived in [1], despite the unavailability of the exact expression of the pgf there. Using the results in [4], we derive exact expressions for the probability generating function and the mean of the packet delay.

The remaining of this paper is organized as follows: In section 2, we describe the queuing model and present the functional equation for the m-dimensional pgf of the state vector. In section 3, the above functional equation is transformed into a new form that is mathematically tractable. Exact expressions for the steady-state marginal pgf of the packet arrival process and the queue length are presented in sections 4 and 5, respectively. In section 6, we use the marginal pgf of the queue length to derive expression for the mean buffer occupancy. In section 7, some results related to the packet delay are presented. In section 8, we illustrate the results of the paper through some numerical results. A summary of the main findings of the papers and recommendations for future research are provided in section 9.

2 Queuing Model and Functional Equation

In this paper, we consider a discrete-time queuing system (figure 1) with infinite buffer capacity, N input links, one output link and a single (FCFS) deterministic server. The time axis is divided into equal length slots and packet transmission is synchronized to occur at the slot boundaries. Here a slot is the time period required to transmit exactly one packet from the buffer, and a message enters the buffer as a train at a fixed rate of one packet per slot. We further assume that each message is composed of a fixed number of *m* packets. In addition, traffic on different input links is assumed to be independent and with the same statistical characteristics.

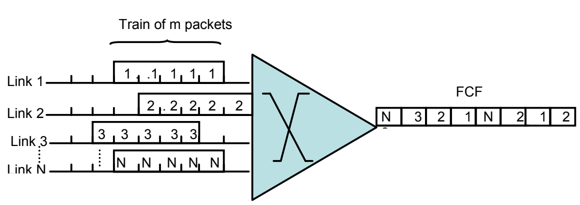

Fig.1. Statistical multiplexer with N input links and *m* packets/train

On any input link, the probability that the first (leading) packet of a message enters the buffer in any given slot is *q* if the first packet of the previous message on this link did not enter the buffer during the previous (m-1) slots and it is 0 otherwise.

Further, let $\{c_j ; j \geq 1\}$ be a series of independent and identical Bernoulli random variables with pgf

$$C(z) = 1-q + qz$$

The queuing model under consideration can be formulated as a discrete-time m-dimensional Markov chain. The state of the system is defined by the state vector $(l_k, a_{1,k}, a_{2,k}, .. a_{m-1,k})$ where l_k is the queue length at the end of slot k and $a_{n,k}$ $(0<n<m)$ is the number of input links having sent the n[th] packet of a message to the buffer in slot k.

Next let:

$$Q_k(z, x_1, x_2, x_3, \ldots x_{m-1}) = E\left[z^{l_k} . x_1^{a_{1,k}} . x_2^{a_{2,k}} . x_{m-1}^{a_{m-1,k}}\right]$$

denote the joint pgf of the system state vector.

This type of statistical multiplexers with correlated train arrivals was modeled in [1] and a functional equation describing the pgf of the system state vector was derived and is given by the following expression [1]:

$$Q_{k+1}(z, x_1, x_2, x_3, \ldots x_{m-1}) = \left[C(x_1 z)\right]^N \left\{ \frac{Q_k\left(z, \frac{x_2 z}{C(x_1 z)}, \frac{x_3 z}{C(x_1 z)}, \ldots \frac{x_{m-1} z}{C(x_1 z)}, \frac{z}{C(x_1 z)}\right) - p_k(0)}{z} + p_k(0) \right\} \quad (1)$$

Where $p_k(0) = \text{Prob}\ (l_k=0)$ is the probability of an empty buffer at the end if the k[th] slot and

$$C(x_1 z) = 1-q + q x_1 z.$$

Applying the classical argument that as $k \to \infty$, the sequences of the functions $Q_{k+1}(z, x_1, x_2, x_3, \ldots x_{m-1})$ and $Q_k(z, x_1, x_2, x_3, \ldots x_{m-1})$ converge to the same limiting function, namely $Q(z, x_1, x_2, x_3, \ldots x_{m-1})$, yields the following functional equation relating the steady-state joint pgf of the system [1]:

$$z.Q(z, x_1, x_2, x_3, \ldots x_{m-1}) = \left[C(x_1 z)\right]^N \left\{ \begin{array}{l} Q\left(z, \frac{x_2 z}{C(x_1 z)}, \frac{x_3 z}{C(x_1 z)}, \ldots \frac{x_{m-1} z}{C(x_1 z)}, \frac{z}{C(x_1 z)}\right) \\ +(z-1)p(0) \end{array} \right\} \quad (2)$$

where p(0) is the steady-state probability of an empty buffer. It is clear from the above approach that it does not allow the derivation of the steady-state join pgf of the system since the functional equation cannot be solved. By letting the arguments of the Q-function on the left hand side and the right hand side of equation (2) be equal to each others, closed-form expression for the mean buffer occupancy was derived in [1], despite the unavailability of the steady-state pgf there. One of the aims of this paper is to show how to tackle such type of functional equations in order to extract an exact expression for the steady-state pgf of the queue length

3. Transforming the Functional Equation into a New Form

In this section, we show how to transform the functional equation (1) of the system into a new form that will lead itself to a solution.

3.1 Proposition 1

Let *u(k)* be defined by the following recurrence relationship:

$$u(0)=1;\ u(1)=C(x_1 z)=1-q+qx_1 z\ ;$$

$$u(k)=u(k-1)\Big|_{x_i=\frac{x_{i+1}z}{1-q+qx_1 z}} \qquad (k\geq 2)$$

where $i=1,2,\ldots m-1$ and where $x_m=1$ for notational convenience.

Then *u(k)* can be expressed in terms of a new sequence *J(k)*, defined as follows:

$$u(k)=\frac{J(k)}{J(k-1)}$$

Where the sequence *J (k)* is defined by the m^{th}-order linear homogeneous "difference" equation:

$$J(k)=(1-q)J(k-1)+qz^m J(k-m) \qquad (k\geq m)$$

With the following *m* initial conditions:

$$J(0)=1$$
$$J(k)=(1-q)J(k-1)+qx_k z^k \qquad (k<m)$$

The proof of the above proposition is readily obtained by induction and will not be given here.

3.2 Proposition 2

Let $\Phi_i(k)$ be defined by the following recurrence relationship:

$$\Phi_i(0)=x_i$$
$$\Phi_i(k)=\Phi_i(k-1)\Big|_{x_i=\frac{x_{i+1}z}{1-q+qx_1 z}} \qquad \forall k\geq 1$$

where $i=1,2,\ldots m-1$ and where $x_m=1$ for notational convenience. Then $\Phi_i(k)$ can be expressed in terms of the sequence *J(k),* above, as follows:

$$\Phi_i(k)=\begin{cases}\dfrac{x_{i+k}z^k}{J(k)} & (k+i<m)\\[2ex] \dfrac{z^{m-i}J(k-m+i)}{J(k)} & (k+i\geq m)\end{cases}$$

The proof of the above proposition is readily obtained by induction and can be found in [5].
Next, we rewrite the functional equation (1) into a more suitable form. First, let $B(k)=[u(k)]^N$, then taking into account the definitions of *B(k)* and $\Phi_i(k)$, as defined before, we can re-write the functional equation (1) as follows:

$$Q_{k+1}(z,x_1,x_2,\ldots,x_{m-1})=B(1)\left\{\frac{Q_k(z,\Phi_1(1),\Phi_2(1),\ldots\Phi_{m-1}(1))-p_k(0)}{z}+p_k(0)\right\}$$

The next theorem presents a major result in the paper, as it allows us to re-write the functional equation of the statistical multiplexer into a mathematically tractable form.

3.3 Theorem 1

The functional equation (1) describing the queuing model under consideration can be written as follows:

$$Q_k(z,x_1,x_2,\ldots x_{m-1})=\frac{\prod_{i=1}^{k}B(i)}{z^k}Q_0(z,\Phi_1(k),\Phi_2(k),\ldots,\Phi_{m-1}(k))$$
$$+(z-1)\sum_{j=1}^{k}\frac{\prod_{i=1}^{j}B(i)}{z^j}p_{k-j}(0)$$

or equivalently:

$$Q_k(z,x_1,x_2,\ldots x_{m-1})=\frac{[J(k)]^N}{z^k}Q_0(z,\Phi_1(k),\Phi_2(k),\ldots,\Phi_{m-1}(k))$$
$$+(z-1)\sum_{j=1}^{k}\frac{[J(j)]^N}{z^j}p_{k-j}(0) \qquad (3)$$

where the summation is taken to be empty for k=0.
The proof of (3) is obtained through simple induction and can be found in [5].
It is interesting to note that the transient joint pgf of the queuing model under consideration, as expressed in (3) is now explicitly defined in terms of the initial joint pgf at slot 0, the sequences $J(k)$ and $\Phi(k)$ as well as the transient probabilities of an empty buffer $p_k(0)$. Further, the Markovian property of the queuing system under consideration implies that the steady-state behavior of the queuing model is independent of the initial conditions, embedded in the $Q_0(z,\Phi_1(k),\Phi_2(k),\ldots,\Phi_{m-1}(k))$ term in equation (3). Therefore, without any loss of generality, we can assume zero initial conditions, where the buffer is initially empty with all links being 'idle'.

Substituting $Q_0(z,x_1,x_2,...x_{m-1})=1$ for zero initial conditions in (3) yields:

$$Q_k(z,x_1,x_2,...x_{m-1})=\frac{[J(k)]^N}{z^k} + (z-1)\sum_{j=1}^{k}\frac{[J(j)]^N}{z^j}p_{k-j}(0) \qquad (4)$$

4. Steady-State Marginal PGF of the Packet Arrival Process

First, we determine the steady-state pgf of the packet arrival process. Let:
$A_k(x_1,x_2,...x_{m-1})=Q_k(1,x_1,x_2,...x_{m-1})=E\left[x_1^{a_{1,k}}.x_2^{a_{2,k}}.x_{m-1}^{a_{m-1,k}}\right]$
be the (m-1)-dimensional transient pgf of the random variables $a_{1,k},a_{2,k},..a_{m-1,k}$. Also, let
$A(x_1,x_2,....x_{m-1})=Q_\infty(1,x_1,x_2,...x_{m-1})=E\left[x_1^{a_1}.x_2^{a_2}.x_{m-1}^{a_{m-1}}\right]$
be the corresponding steady state pgf, where a_r ($1\le r\le m-1$) denotes the number of links having sent their r^{th} packet of a message to the buffer in an arbitrary slot in steady state.
From (4):

$$A_k(x_1,x_2,...x_{m-1})=\left[\hat{J}(k)\right]^N$$

where $\hat{J}(k)=J(k)\big|_{z=1}$.
To derive the steady-state pgf $A_\infty(x_1,x_2,...x_{m-1})=\lim_{k\to\infty}\left[\hat{J}(k)\right]^N$, we will make use of the following proposition:

4.1 Proposition 3:

Let us define the following transform:
$G(w)=\sum_{k=0}^{\infty}J(k)w^k \quad ; |w|\le 1$, where the sequence *J(k)* is as previously defined in proposition 1. Then:

$$G(w)=-\frac{1+q\sum_{k=1}^{m-1}x_k z^k w^k}{qz^m w^m+(1-q)w-1} \qquad (5)$$

The proof to the above proposition is readily obtained by simple transformation techniques and can be found in [5].
Next, we determine the steady-state joint pgf of $a_1,a_2,..a_{m-1}$ by applying the final value theorem to (5)

$$A(x_1,x_2,....x_{m-1})=\left[\lim_{w\to1^-}(1-w)\hat{G}(w)\right]^N \qquad (6)$$

where

$$\hat{G}(w)=G(w)\big|_{z=1}=-\frac{1+q\sum_{k=1}^{m-1}x_k w^k}{qw^m+(1-q)w-1}.$$

From the above, we note that $\lim_{w\to1^-}(1-w)\hat{G}(w)$ is zero, except at the single singularity where the denominator of $\hat{G}(w)$ is equal zero at w=1. Hence :

$$\lim_{w\to1^-}(1-w)\hat{G}(w)=\frac{1+q\sum_{k=1}^{m-1}x_k}{mq+(1-q)}$$

which yields the following explicit expression of the steady-state pgf of the random variables $a_1,a_2,..a_{m-1}$:

$$A(x_1,x_2,....x_{m-1})=\left[\frac{1+q\sum_{k=1}^{m-1}x_k}{1+(m-1)q}\right]^N \qquad (7)$$

The packet arrival process in the steady-state, is explicitly characterized the m-dimensional joint pgf, $A(x_1,x_2,....x_{m-1},x_m)$, of all random variables $a_1,a_2,..a_{m-1}$, a_m, which, from (7), is given by:

$$A(x_1,x_2,....x_{m-1},x_m)=\left[\frac{1-q+q\sum_{k=1}^{m}x_k}{1+(m-1)q}\right]^N \qquad (8)$$

Next, let $A_r(x_r)$ denotes the marginal pgf of a_r ($1\le r\le m$. This marginal pgf can be derived from the joint pgf (8) by setting x_k=1 for ($1\le k\le m$ and $k\ne r$), giving:

$$A_r(x_r)=\left[1-\frac{q}{1+(m-1)q}+\frac{q}{1+(1-m)q}x_r\right]^N \qquad (9)$$

From (9), the steady-state arrival rate of the packets to the system is given by $\frac{Nqm}{1+(m-1)q}$. Since the service rate is one packet/slot, the load of the system is:

$$\rho=\frac{Nqm}{1+(m-1)q} \qquad (10)$$

and for a stability, we require that $\rho<1$.
Note that expression 9, describing the arrival process and expression (10) for the load of the system are equivalent to those derived in [1]. The only difference is that while these derivations in [1] were based on probabilistic arguments related to packet arrivals, our results were derived directly from the joint pgf of the system.

5. Steady-State Marginal PGF of the Queue Length

In this section, we determine the steady-state marginal pgf of the queue length. Let $P_k(z)=Q_k(z,x_1,x_2,...x_{m-1})\big|_{x_1=x_2=...x_{m-1}=1}$ denotes the marginal pgf of the buffer occupancy at the end of the k^{th} slot, assuming zero initial conditions. From (4):

$$P_k(z)=\frac{\left[\widetilde{J}(k)\right]^N}{z^k}+(z-1)\sum_{j=1}^{k}\frac{\left[\widetilde{J}(j)\right]^N}{z^j}p_{k-j}(0) \quad (11)$$

where $\widetilde{J}(k)=J(k)\Big|_{x_1=x_2=...x_{m-1}=1}$.

In order to extract the steady-state marginal pgf *P(z)* = $\lim_{k\to\infty}P_k(z)$ of the queue length from the above expression, we will make use of the following proposition:

5.1 Proposition 4:

The function $\widetilde{J}(k)=J(k)\Big|_{x_1=x_2=...x_{m-1}=1}$, appearing in (11) is given by the following formula:

$$\widetilde{J}(k)=\sum_{i=1}^{m}C_i\lambda_i^{-k} \quad (12)$$

where:

$$C_i=\frac{(1-q)(z-1)\lambda_i}{(1-\lambda_i z)\left[(m-1)(1-q)\lambda_i-m\right]} \quad (13)$$

and λ_i's (*i=1,2, ..m*) are the *m* distinct roots of the characteristic equation:

$$q(z\lambda)^m+(1-q)\lambda-1=0 \quad (14)$$

The proof of the above proposition is readily obtained via simple transform techniques and can be found in [5].

From (14), it is obvious that one of the roots has the property that $\lambda\big|_{z=1}=1$. This particular root is thereafter denoted by λ_m

Next let us define the following transforms ($|w|\le 1$):

$$P(z,w)=\sum_{k=0}^{\infty}P_k(z)w^k \;\; ; \;\; P(w)=\sum_{k=0}^{\infty}p_k(0)w^k \quad (15)$$

Now substituting $P_k(z)$ from (11) into *P(z,w),* as defined in (15):

$$P(z,w)=\sum_{k=0}^{\infty}\frac{\left[\widetilde{J}(k)\right]^N}{z^k}w^k+(z-1)\sum_{k=1}^{\infty}\sum_{j=1}^{k}\frac{\left[\widetilde{J}(j)\right]^N}{z^j}p_{k-j}(0)w^k$$

Interchanging the order of summations in the second term and recognizing *P(w)* defined in (15), yields

$$P(z,w)=\sum_{k=0}^{\infty}\frac{\left[\widetilde{J}(k)\right]^N}{z^k}w^k+(z-1)P(w)\sum_{j=1}^{\infty}\frac{\widetilde{J}(j)^N}{z^j}w^j$$

Substituting for $\widetilde{J}(k)$ from (12) into the above yields:

$$P(z,w)=\sum_{k=0}^{\infty}\frac{\left[\sum_{l=1}^{m}C_l\lambda_l^{-k}\right]^N}{z^k}w^k+(z-1)P(w)\sum_{j=1}^{\infty}\frac{\left[\sum_{i=1}^{m}C_i\lambda_i^{-j}\right]^N}{z^j}w^j$$

Substituting for the Multinomial expansion and interchanging the order of the summation,

$$P(z,w)=\sum_{n_1+n_2+...+n_m=N}\sum_{k=0}^{\infty}\frac{N!}{n_1!.n_2!...n_m!}\left[\prod_{i=1}^{m}\left(C_i\lambda_i^{-k}\right)^{n_i}\right]\left(\frac{w}{z}\right)^k$$
$$+(z-1)P(w)\sum_{n_1+n_2+...+n_m=N}\sum_{j=1}^{\infty}\frac{N!}{n_1!.n_2!...n_m!}\left[\prod_{i=1}^{m}\left(C_i\lambda_i^{-j}\right)^{n_i}\right]\left(\frac{w}{z}\right)^j$$

Separting the inner summations, and simplifying gives:

$$P(z,w)=\sum_{n_1+n_2+...+n_m=N}\frac{N!}{n_1!.n_2!...n_m!}\cdot\frac{z.\prod_{i=1}^{m}(C_i)^{n_i}}{z-w\prod_{i=1}^{m}\lambda_i^{-n_i}}$$
$$+(z-1)P(w).w\sum_{n_1+n_2+...+n_m=N}\frac{N!}{n_1!.n_2!...n_m!}\cdot\frac{\prod_{i=1}^{m}\left(\frac{C_i}{\lambda_i}\right)^{n_i}}{z-w\prod_{i=1}^{m}\lambda_i^{-n_i}}$$

Next, we determine the steady-state PGF of the queue length by applying the final value theorem to the last expression,

$$P(z)=\lim_{w\to 1^-}(1-w)P(z,w)$$

In *P(z,w)* the first term is transient while the second term leads us to the steady-state, thus we have:

$$P(z)=\lim_{w\to 1^-}(1-w)P(w)(z-1)w\sum_{n_1+n_2+...+n_m=N}\frac{N!}{n_1!.n_2!...n_m!}\cdot\frac{\prod_{i=1}^{m}\left(\frac{C_i}{\lambda_i}\right)^{n_i}}{z-w\prod_{i=1}^{m}\lambda_i^{-n_i}}$$

Since

$$p_\infty(0)=\lim_{w\to 1^-}(1-w)P(w)=1-\rho,$$

Substituting this into *P(z)* gives the final expression for the steady-state PGF of the queue length,

$$P(z)=(1-\rho)(z-1)\sum_{n_1+n_2+...+n_m=N}\frac{N!}{n_1!.n_2!...n_m!}\cdot\frac{\prod_{i=1}^{m}\left(\frac{C_i}{\lambda_i}\right)^{n_i}}{z-\prod_{i=1}^{m}\lambda_i^{-n_i}} \quad (16)$$

Though the stability condition of the system was already determined in section 4.1, it can also be determined from the normalization condition $P(z)\big|_{z=1}=1$, as follows:

First recall that

$$\frac{C_i}{\lambda_i}\bigg|_{z=1}=\frac{(1-q)(z-1)}{(1-\lambda_i z)\left[(m-1)(1-q)\lambda_i-m\right]}\bigg|_{z=1}$$ is zero for all i's , except at the singularity corresponding the root λ_i which satisfies the condition $\lambda_i\Big|_{z=1}=0$. This particular root was previously denoted by λ_m. Then it is convenient to re-write the steady-state PGF of the queue length as follows:

$$P(z)=(1-\rho)(z-1)\left[F(z)+\frac{G(z)}{z-H(z)}\right]$$

where:

$$F(z)=\sum_{\substack{n_1+n_2+\ldots+n_m=N \\ n_1.n_2.n_3..n_{m-1}\neq 0}} \frac{N!}{n_1!.n_2!...n_m!}.\frac{\prod_{i=1}^{m}\left(\frac{C_i}{\lambda_i}\right)^{n_i}}{z-\prod_{i=1}^{m}\lambda_i^{-n_i}}$$

$G(z)=\left(C_m\lambda_m^{-1}\right)^N$ and $H(z)=\lambda_m^{-N}$

or equivalently

$$P(z)[z-H(z)]=(1-\rho)(z-1)[(z-H(z))F(z)+G(z)] \quad (17)$$

Note that in the above expression F(1)=0, H(1)=1, and G(1) =1. Differentiating both sides of the above equation with respect to *z* and setting *z=1* yields $\rho=H'(1)=-N.\lambda'_m(1)$. From (14) it is easy to verify that $\lambda'_m(1)=\frac{-mq}{1+q(m-1)}$ and therefore $\rho=\frac{Nqm}{1+(m-1)q}$, in accordance with (10).

Note that for small values of *m*, more explicit expressions for the PGF of the queue length can be obtained from (14) and (16). For instance, for *m*=1 we get :

$$P(z)=(1-\rho)(z-1)\frac{(qz+1-q)^N}{z-(qz+1-q)^N}$$

Similarly, for *m*=2, we get:

$$P(z)=(1-\rho)(z-1)\sum_{n_1=0}^{N}\binom{N}{n_1}.\frac{\left(C_1\lambda_1^{-1}\right)^{n_1}.\left(C_2\lambda_2^{-1}\right)^{N-n_1}}{z-\lambda_1^{-n1}.\lambda_2^{-(N-n1)}}$$

where C_1 and C_2 are as defined in (13), and :

$$\lambda_1=\frac{-1+q-\sqrt{(1-q)^2+4qz^2}}{2qz^2};$$

$$\lambda_2=\frac{-1+q+\sqrt{(1-q)^2+4qz^2}}{2qz^2}$$

6. Mean Buffer Occupancy

Let $\overline{N}$ denote the mean buffer occupancy in steady-state. By differentiating (17) twice with respect to z and setting z=1 and the resulting expression, we get:

$$\overline{N}=\frac{H''(1)}{2[1-H'(1)]}+G'(1) \quad (18)$$

where:

$$H''(1)=N(N+1)[\lambda_m'(1)]^2-N\lambda_m''(1)$$
$$=\frac{(qmn)^2-mN(mq^2-(1-q)^2)}{(1+q(m-1))^2}-\frac{mN(1-q)^2}{(1+q(m-1))^3}$$

$$G'(1)=\frac{3qmN}{2(1+q(m-1))}-\frac{qm^2N}{2(1+q(m-1))^2};$$

$$H'(1)=\rho$$

The above expression (18) can also be expressed in terms of the load ρ of the system to yield:

$$\overline{N}=\rho+\frac{(N-1)}{N}\frac{m\rho^2}{2(1-\rho)}-\frac{(N-1)}{N^2}\frac{(m-1)\rho^3}{2(1-\rho)} \quad (19)$$

The above expression is equivalent to the corresponding result in [1] without the availability of the PGF of the queue length. Higher moments of the buffer occupancy can also be obtained by successive differentiation of (17) but will not be given here.

7. Steady-State packet Delay

Let D(z) be the PGF of the delay of an arbitrary packet in steady-state. This delay represents the number of slots between the end of the packet's arrival slot and the end of the packet's transmission slot. In [4] it was shown that for any discrete-time single-server queuing system, with FCFS queuing discipline and constant service time of one slot, D(z) is related to the PGF of the queue length, P(z), through the following relationship:

$$D(z)=\frac{P(z)-(1-\rho)}{\rho} \quad (20)$$

The above result will be most useful in the derivation of the moment of packet delay. In particular, let $\overline{d}$ denote the average packet delay, then from (19) and (20):

$$\overline{d}=\frac{\overline{N}}{\rho}=1+\frac{(N-1)}{N}\frac{m\rho}{2(1-\rho)}-\frac{(N-1)}{N^2}\frac{(m-1)\rho^2}{2(1-\rho)} \quad (21)$$

8. Numerical Results

In this section, we illustrate our solution method through some numerical examples, illustrated in figures 2 and 3 below. First, the overflow probability (packet loss rate) due to a finite buffer size (*S*) is often approximated by the probability that the buffer occupancy in an infinite buffer system exceeds the proposed buffer size (i.e. Prob ($l > S$)). In our case, exact probabilities can be computed by observing that the required probabilities correspond to the coefficients of Z^S, in the polynomial $\frac{1-P(z)}{1-z}$.

These probabilities, derived from the Taylor series expansion, using Maple™ [6] computational software, are illustrated in figure 2, below. As may be seen from figure 2, and as expected, the probability of overflow gets larger as the correlation of the packet arrival process increases. Note that the probability of overflow curve for m=1 corresponds to the uncorrelated case.

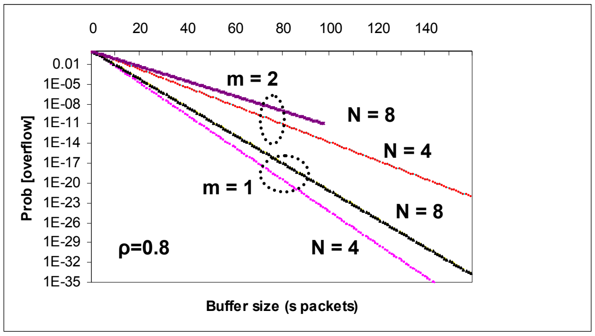

Fig.2 Probability of overflow versus buffer size, ρ=0.8 Erlang, N=4,8 and m=1,2

Figure 3 also presents some results related to the mean packet delay. We also note from this figure the negative effect of correlation (increasing *m*) on the average packet delay.

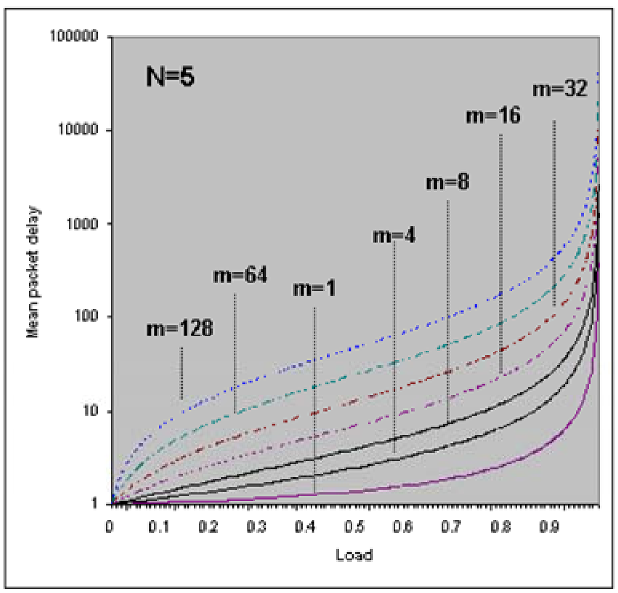

Fig.3 Mean packet delay versus load ρ, for N=5 and m=1,4,8,16,32,64, and 128

9. Conclusions and Suggestions for Further Research

In this paper, we have carried an exact queuing analysis of a statistical multiplexer with a finite number of input links and whose arrival process consists of a train of a fixed number of fixed-length packets. By means of a generating functions approach, coupled with functional transformation techniques, we were able to extract an exact expression for the steady-state probability generating functions (pgfs) of the queue length and packet arrivals. From the pgf of the queue length, several performance measures such as mean buffer occupancy, mean packet delay and buffer overflow probabilities were derived. The transform approach used in the present analysis provides a general framework under which similar types of functional equations, arising in the performance analysis of statistical multiplexers, can be tackled. Unlike other queuing methods such as those based on matrix geometric and spectral decomposition approaches, our solution provides more explicit results, which are not in general matrix form. Further, the results presented in this paper have diverse applications in the buffer dimensioning, congestion control, and resource management of the queuing model under consideration. This work can be further explored in many directions. For example, since the inversion of the steady-state pgf of the queue length is not trivial, tight upper-bounds for the tail distribution of the buffer occupancy ought to be derived. This is left for future research.

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[6] Maple is a registered trademark of Maplesoft (http://www.maplesoft.com/)